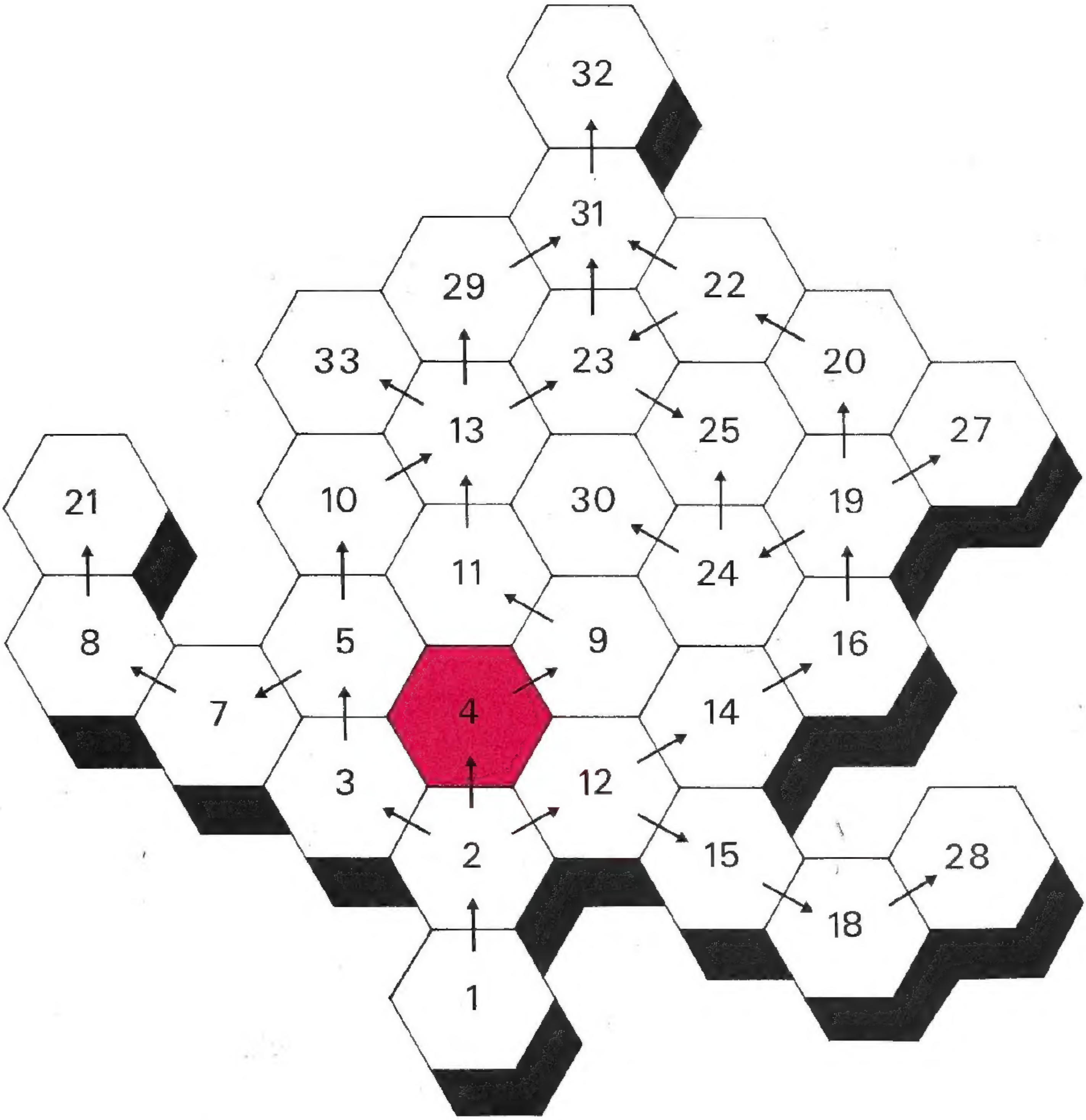
M201 4

THE OPEN UNIVERSITY

Mathematics: A Second Level Course

Linear Mathematics Unit 4

Differential Equations I



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Mathematics: A Second Level Course

Linear Mathematics *Unit 4*

DIFFERENTIAL EQUATIONS I

Prepared by the Linear Mathematics Course Team

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1.2

Contents

Set Books (Paperback Editions)

D. L. Kreider, R. G. Kuller, D. R. Ostberg and F. W. Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E. D. Nering, *Linear Algebra and Matrix Theory* (John Wiley, 1970).

It is essential to have these books; the course is based on them and will not make sense without them.

Conventions

Before working through this correspondence text make sure you have read *A Guide to the Linear Mathematics Course*. Of the typographical conventions given in the Guide the following are the most important.

The set books are referred to as:

K for *An Introduction to Linear Analysis*

N for *Linear Algebra and Matrix Theory*

All starred items in the summaries are examinable.

References to the Open University Mathematics Foundation Course Units (The Open University Press, 1971) take the form *Unit M100 3, Operations and Morphisms*.

4.0 INTRODUCTION

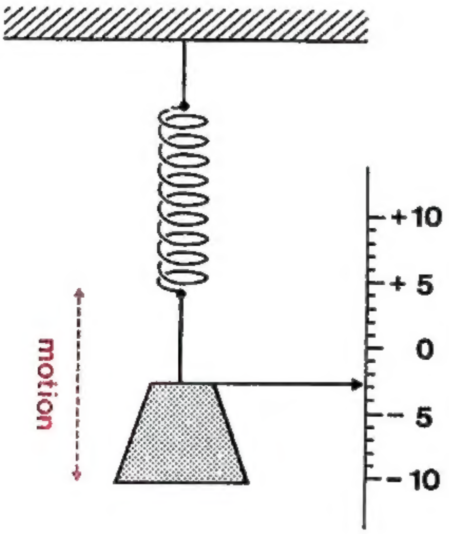
One of the principal applications of linear mathematics is to linear differential equations. You have seen already in the Foundation Course (*Units M100 24* and *31*, *Differential Equations I and II*) how linear differential equations can arise in the theories of population growth and mechanical vibrations. Another application is to electric circuits; it is the subject of the television component of this unit. There are many other applications, mostly in physical science and the technology based on it; you will meet some of these later in the course.

In general, differential equations are harder to solve than algebraic equations because the solutions are functions rather than numbers. However, just as an algebraic equation is simpler to solve when it is linear, so too a differential equation is easier to solve when it is linear; because it is then a *linear problem* (in the sense defined on page N63), and the solution set then has a particularly simple algebraic structure. For example, in the differential equation

$$f'' + f = 0 \qquad (\text{domain of } f \text{ is } R_0^+)$$

which we used in the Foundation Course (*Unit M100 31*, *Differential Equations II*) to model a mass–spring system, the solution set consists of all functions obtainable by giving real values to A and B in the formula

$$f: t \longmapsto A \cos t + B \sin t \qquad (t \in R_0^+).$$



In other words, the solution set is a vector space, consisting of all linear combinations of the functions cos and sin; in the notation of *Unit 1*, *Vector Spaces*, it is the set spanned by cos and sin, i.e.

$$\langle \cos, \sin \rangle.$$

Why is the solution set of the differential equation a vector space? The reason is that the equation can be written in the form

$$L(f) = 0$$

where L is a linear transformation of some vector space of differentiable functions; the solution set of such an equation is the kernel of L, and we know already (page N31) that the kernel of any linear transformation is a vector space. In the present case the linear transformation in question is defined by

$$L(f) = f'' + f$$

with a suitable domain.

In this unit we will investigate the ideas sketched above and generalize them to yield more general linear differential equations (i.e. differential equations that are linear problems). In particular, we shall see how to reduce the problem of solving *any* linear differential equation of first order to the evaluation of integrals.

In order to be able to complete the solutions to these and other problems involving integration, you will need to use some techniques which are based on the rules of integration by substitution and integration by parts (see *Unit M100 13, Integration II*).

It is not the purpose of this course to teach these techniques, but the main results are collected in the supplementary handbook, *Techniques of Integration* (referred to as **TI**) to which you should refer when necessary.

4.1 LINEAR DIFFERENTIAL OPERATORS

4.1.0 Introduction

In solving the differential equation

$$f'' + f = 0$$

we are led to consider the linear transformation L defined by

$$L(f) = f'' + f \qquad (f \in \text{some domain})$$

Such a linear transformation, mapping any function in its domain to a linear combination of itself and one or more of its derived functions is called a *linear differential operator*. We shall give a more precise definition later, but before we can do so, it is necessary to decide what set of functions to use as the domain for such an operator.

4.1.1 The D Operator

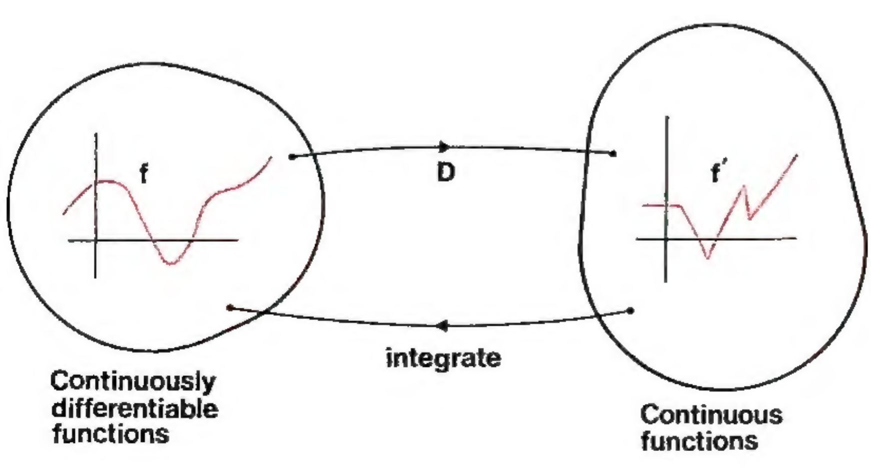
The simplest linear differential operator is the differentiation operator

$$D: f \longmapsto f' \qquad (f \in \text{some domain of } D).$$

We know that D is a linear transformation from previous work (*Unit 2*, sub-section 2.1.5, Exercise 3) but what about its domain? The obvious choice for the domain of D is the set of all real functions (functions whose domain and codomain are R or subsets of R) that are differentiable everywhere in their domain. (This means that $f'(x)$ exists for all x in the domain of f.) We shall call such functions *real differentiable functions*. The obvious choice is not always the best, however, and here two departures from the obvious are desirable. First, in order to solve equations of the form

$$Df = (\text{given function in image set of } D)$$

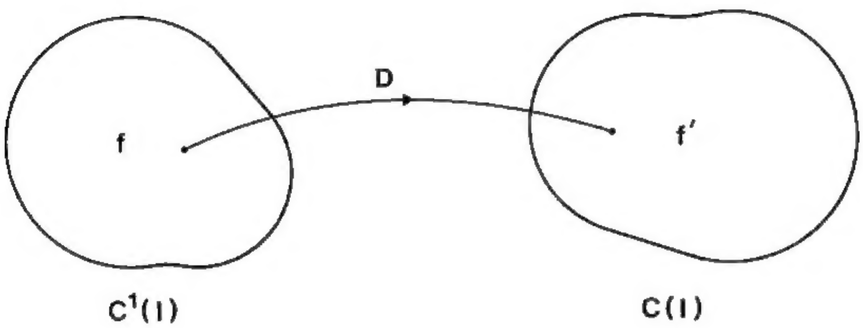
we would like to make use of the Fundamental Theorem of Calculus. This theorem (as stated in *Unit M100 13, Integration II*) tells us that differentiation can be reversed by integration *provided the function to be integrated is continuous*. So, to make sure that the fundamental theorem will apply, we would like all the functions in the image set to be continuous. We can do this if we re-define the domain of D to consist of the real differentiable functions f, such that *f' is continuous*. The graph of such a function, f, is not only continuous but is also *smooth*: the slope of the tangent varies continuously. Such functions are said to be *continuously differentiable*. (Perhaps you are wondering why we require f to be *continuously* differentiable: surely the derived function of a differentiable function is necessarily continuous? Unfortunately this is not so; there are "pathological" differentiable functions whose derived functions are not continuous. An example is given in the Appendix.)



A second departure from the obvious is to specify the type of subset we will admit as the domain of our function f: so far we have implied, rather imprecisely, that any subset of R would do. The Fundamental Theorem of Calculus works for real continuously differentiable functions, provided the domain has no gaps. A subset of R without gaps is called an *interval*; for example, R and R^+ are intervals. You have already met notations such as

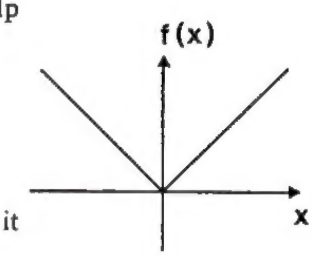
$$[a, b] \text{ for the interval } \{x: a \leqslant x \leqslant b\}$$

in the Foundation Course. We use the notation $\mathcal{C}(I)$ for the set of all real functions continuous in some interval I; the particular case $\mathcal{C}[a, b]$ was used in *Unit 1, Vector Spaces* as an example of a vector space (see page K3). We also use the notation $\mathcal{C}^1(I)$ for the set of all functions continuously differentiable in the interval I; this set is a suitable domain for the linear differential operator D.



Example

We have introduced a few words and notations in this section: to help them sink in we look at an example.



Let f be the modulus function

$$f: x \longmapsto \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases} \qquad (x \in R)$$

Clearly f is continuous: so $f \in \mathcal{C}(R)$. But f is not differentiable because it has no derivative at 0.

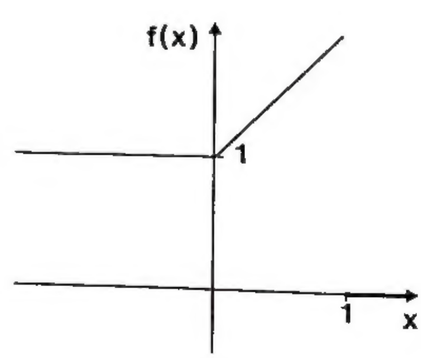
Suppose we restrict the domain of f to $[1, 3]$, say. Then $f \in \mathcal{C}[1, 3]$ and

$$f': x \longmapsto 1 \qquad (x \in [1, 3])$$

So f' is continuous and, hence, f is continuously differentiable in $[1, 3]$; i.e. $f \in \mathcal{C}^1[1, 3]$.

As further examples, the sine and cosine functions belong to $\mathcal{C}(R)$ and $\mathcal{C}^1(R)$.

Exercises



1. Let f be the function defined by

$$x \longmapsto \begin{cases} 1 & (x \leqslant 0) \\ 1 + x & (0 < x \leqslant 1) \\ 0 & (x > 1). \end{cases}$$

   Which of the following statements are true?

   (a) $f \in \mathcal{C}(R)$ (b) $f \in \mathcal{C}[-1, 1]$

   (c) $f \in \mathcal{C}^1[-1, 1]$ (d) $f \in \mathcal{C}^1[2, 4]$.

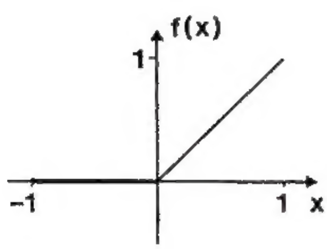
2. Give an example of a function not already given that is continuous but not continuously differentiable in the interval $[-1, 1]$.
3. Is it true or false that $\mathcal{C}^1[a, b] \subset \mathcal{C}[a, b]$?

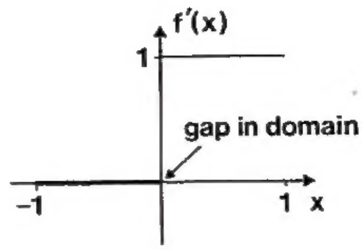
4. Verify that $C^1[a, b]$ is a real vector space. (*Hint* Look at the subspace criterion at the bottom of page K12. We discussed this in *Unit 1*.)

Solutions

1. (a) FALSE (b) TRUE (c) FALSE (d) TRUE
2. One example is the function

$$f: x \longmapsto \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 \end{cases} \qquad (x \in [-1, 1])$$





The function f is continuous, since its graph has no breaks, but it is not differentiable at 0, since

$$\frac{f(0+h)-f(0)}{h} = \begin{cases} +1 & \text{if } h > 0 \\ 0 & \text{if } h < 0 \end{cases}$$

so that $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$ does not exist (see *Unit M100 7*, p. 30).

(If you were unable to find an example yourself, try and find another one, now. It is a good habit to adopt for this type of exercise.)

3. True; for every differentiable function is continuous. *Proof* If the function f is differentiable at a then the limit in the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

exists.

To prove that f is continuous at a, we have to prove that

$$\lim_{x \to a} f(x)$$

exists and is equal to $f(a)$.

We can write

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

whence

$$\begin{aligned} \lim_{x \to a} f(x) - f(a) &= \lim_{h \to 0} f(a+h) - f(a) \\ &= \lim_{h \to 0} h \times \left\{\frac{f(a+h)-f(a)}{h}\right\} \\ &= \left\{\lim_{h \to 0} h\right\} \times \left\{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}\right\}^* \\ &= 0 \times f'(a) \\ &= 0 \end{aligned}$$

4. The solution is given in Example 3 on page K13. Be sure to read it.

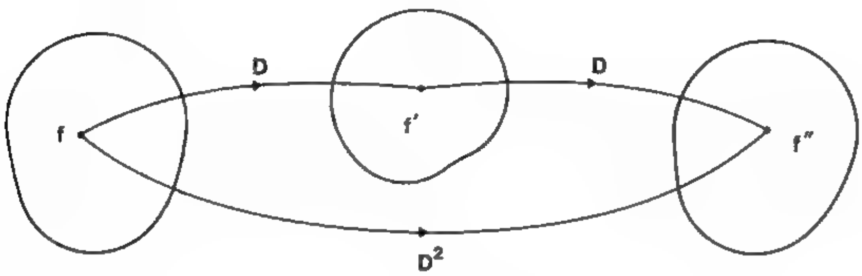
* This step follows from the fact that if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist then $\lim_{x \to a} (f(x)\, g(x))$ exists and is equal to their product.

4.1.2 Powers of the D Operator

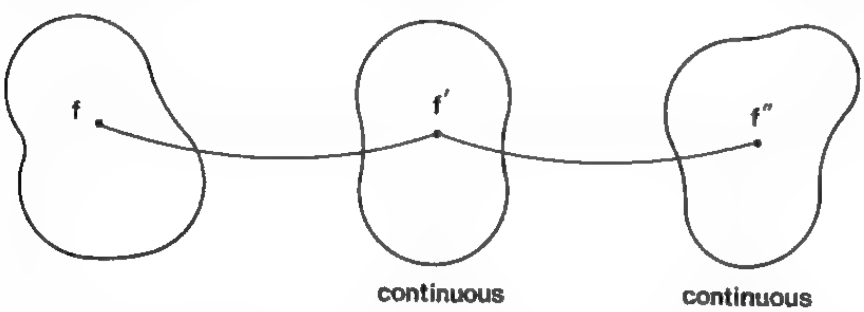
For differential equations of the second and higher orders we shall need, in addition to D, operators giving the second and higher derived functions of a function. The simplest of these is the operator mapping a function f to its second derived function f''; we saw in the Foundation Course that this operator is the composition of D with itself, i.e.

$$D \circ D(f) = D(D(f)) = f''.$$

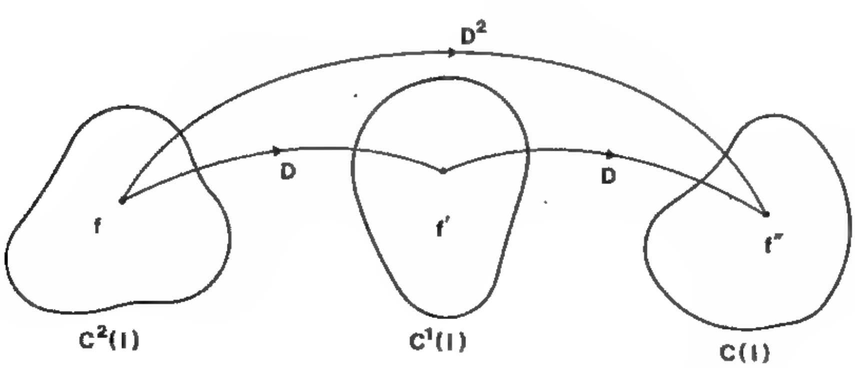
It is usually abbreviated to D^2.



To define D^2 properly we should specify its domain, as we did for D. Once again, if we are to invert the operator D^2 using the fundamental theorem twice, we shall want the functions we integrate to be continuous —that is, we want both f'' and f' to be continuous.



In fact, it is sufficient to require f'' to be continuous; for then f' is continuously differentiable, and any differentiable function is continuous (see Exercise 3 of sub-section 4.1.1). Any function f whose second derived function f'' is continuous is said to be ***twice continuously differentiable***. We denote the set of twice continuously differentiable functions in some



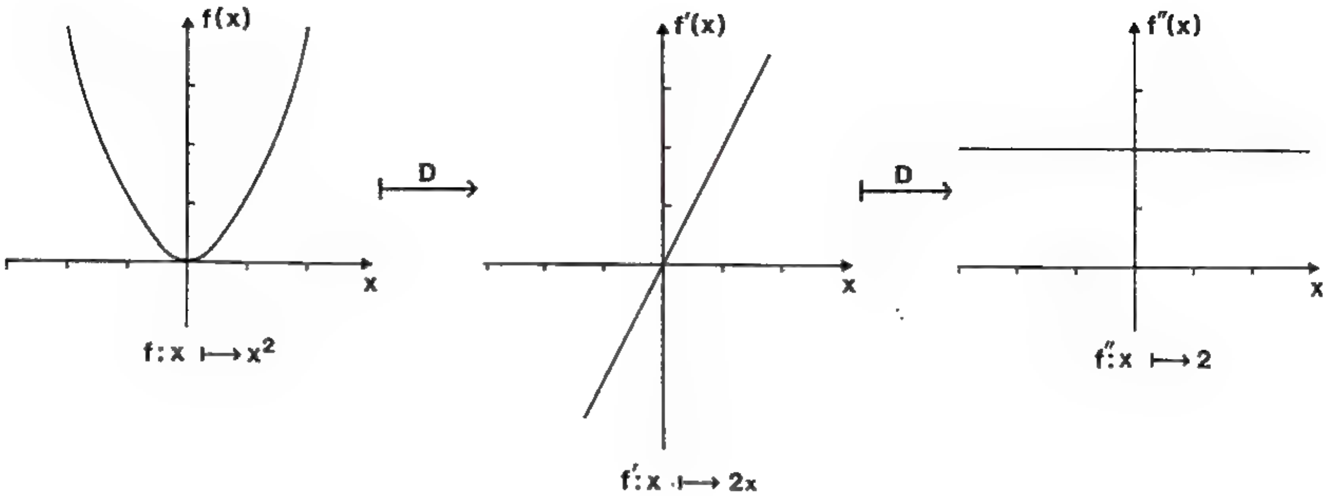
interval I by $\mathcal{C}^2(I)$. Thus the domain of D^2 may be taken as $\mathcal{C}^2(I)$, with I any specified interval of R. (We proved that D^2 is a linear transformation in Exercise 3 of sub-section 5.3 of *Unit 2*.)

Example

The function

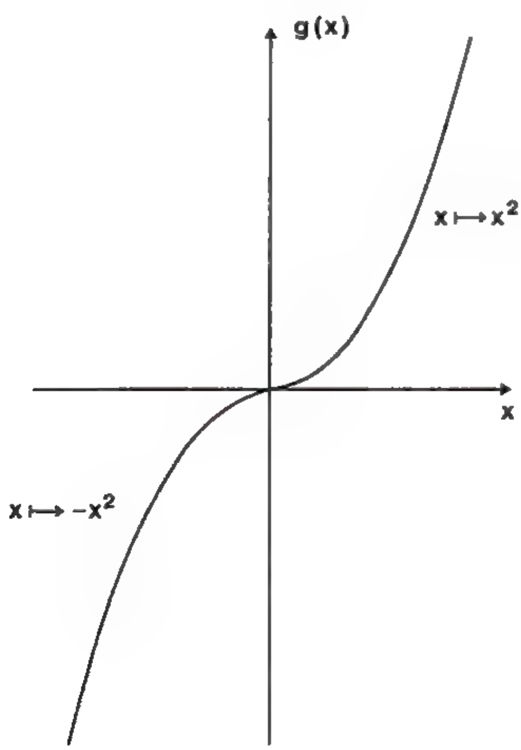
$$f: x \longmapsto x^2 \qquad (x \in R)$$

belongs to $\mathcal{C}^2(I)$ for any interval I.



On the other hand, consider the function

$$g: x \longmapsto \begin{Bmatrix} x^2 & x \geqslant 0 \\ -x^2 & x < 0 \end{Bmatrix} \qquad (x \in R)$$



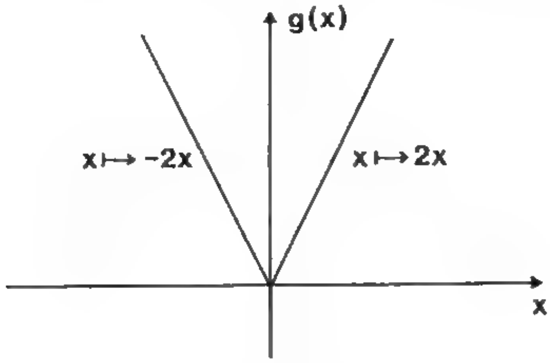
graph of g

This function is differentiable everywhere: the only point requiring investigation is 0. The graph would seem to indicate that there is a derivative

at 0, but one could also verify this formally by considering

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$$

for h positive and h negative: in either case the limit is zero. Also g' is continuous: therefore, $g \in \mathcal{C}^1(R)$.



graph of g′

Clearly g'' is defined except at 0. So, for instance, we have

$$g \notin \mathcal{C}^2(R), g \notin \mathcal{C}^2[-1, 1], g \in \mathcal{C}^2[1, 3].$$

In a similar way we can define linear transformations D^3, D^4, etc.; their domains can be taken as $\mathcal{C}^3(I)$, $\mathcal{C}^4(I)$, etc., for any I where $\mathcal{C}^3(I)$ is the set of all functions three times continuously differentiable in the interval I, and so on.*

For example, $f \in \mathcal{C}^n(R)$ means that the nth derived function of f is continuous at all points of the real line.

It is an immediate consequence of the definition that each of these spaces, $\ldots \mathcal{C}^3(I), \mathcal{C}^2(I), \mathcal{C}^1(I), \mathcal{C}(I)$, is a subset (in fact a subspace) of its predecessor; in symbols we write

$$\mathcal{C}(I) \supset \mathcal{C}^1(I) \supset \mathcal{C}^2(I) \supset \mathcal{C}^3(I) \supset \cdots$$

Exercises

1. (i) Is the function $x \longmapsto x^2 + x^3$, $(x \in R)$ an element of

   (a) $\mathcal{C}^1[0, 1]$, (b) $\mathcal{C}^3[0, 1]$, (c) $\mathcal{C}^{25}[0, 1]$?

   (ii) Is the function $x \longmapsto |x|x^2$, $(x \in R)$ an element of

   (a) $\mathcal{C}^1(R)$, (b) $\mathcal{C}^2(R)$, (c) $\mathcal{C}^3(R)$?

2. Given an example of a function belonging to $\mathcal{C}^n(R)$ for every n.
3. Give an example of a function belonging to $\mathcal{C}^1[-1, 1]$ but not $\mathcal{C}^2[-1, 1]$. (*Hint* Can you use Solution 2 of sub-section 4.1.1?)
4. Give an example of a function belonging to $\mathcal{C}^n[-1, 1]$ but not to $\mathcal{C}^{n+1}[-1, 1]$.

Solutions

1. (i) (a) Yes, (b) Yes, (c) Yes.

   (ii) An alternative definition of the function is

$$x \longmapsto \begin{cases} x^3 & x \geqslant 0 \\ -x^3 & x < 0 \end{cases} \qquad (x \in R)$$

* Strictly, the domain of a function is an integral part of the function, so that D^n can only have one domain. But we assume that D^n is given a specific domain $\mathcal{C}^n(I)$ in any context, and we do not invent a new symbol at each occurrence.

and this is now similar to the example we discussed above.

The derived function is

$$x \longmapsto \begin{cases} 3x^2 & x \geqslant 0 \\ -3x^2 & x < 0 \end{cases} \qquad (x \in R)$$

and we know, from the example, that this latter function belongs to $\mathcal{C}^1(R)$. Therefore the original function belongs to $\mathcal{C}^2(R)$.

The answers are, therefore,

(a) Yes, (b) Yes, (c) No.

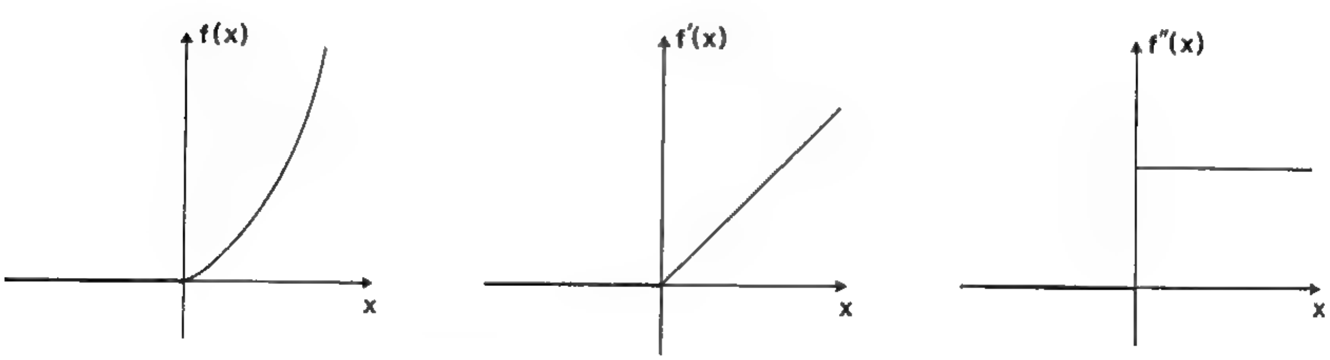
2. The zero function is the simplest example, though perhaps too trivial for some tastes. $x \longmapsto x^2 + x^3$ is also an example. Among the many others are constant functions, powers ($x \longmapsto x^n$ with $n \in Z^+$), polynomial functions, the exponential function, the sine and cosine functions. The space of functions with this property is denoted by $\mathcal{C}^\infty(I)$. We may loosely say that such functions are "infinitely" differentiable on I.
3. If f is the required function, the condition to be satisfied is equivalent to the statement that f' is continuous but not continuously differentiable. By solution 2 of sub-section 4.1.1, one such function is given by

$$f'(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 \end{cases} \qquad (x \in [-1, 1])$$

Since this f' is continuous we can obtain f by integration; we obtain

$$f = \int f' = \begin{cases} x \longmapsto \frac{1}{2}x^2 + k & \text{if } x > 0 \\ x \longmapsto k & \text{if } x \leqslant 0 \end{cases} \qquad (x \in [-1, 1])$$

where k is any constant. The figure depicts the case k=0.



There are many other solutions to this exercise; e.g.

$$f(x) = \begin{cases} x^2 & 1 \geqslant x > 0 \\ 0 & -1 \leqslant x \leqslant 0 \end{cases}$$

4. We can generalise the method used in Solution 3: $f^{(n)}$, the nth derived function of f, is to be continuous but not continuously differentiable; so we may take $f^{(n)}(x)$ to be x if $x > 0$ and 0 if $x \leqslant 0$. Integrating n times we obtain

$$f = \begin{cases} x \longmapsto \dfrac{x^{n+1}}{(n+1)!} + k & \text{if } x > 0 \\ x \longmapsto k & \text{if } x \leqslant 0 \end{cases} \qquad (x \in [-1, 1])$$

where k is any constant.

Once again there are many alternative answers.

This example shows that each of the inclusion relations in the chain mentioned earlier

$$\mathcal{C}(I) \supset \mathcal{C}^1(I) \supset \mathcal{C}^2(I) \supset \cdots,$$

is proper; i.e. for each $n \in Z^+$ there are functions in $\mathcal{C}^n(I)$ which are not in $\mathcal{C}^{(n+1)}(I)$.

4.1.3 Polynomials in D

The linear transformation we considered at the beginning of this unit,

$$L: f \longmapsto f'' + f \qquad (f \in \text{domain of } L)$$

can be expressed as the sum of two simpler linear transformations

$$f \longmapsto f''$$

and

$$f \longmapsto f$$

with the same domain. (The sum of linear transformations was discussed in *Unit 2*, sub-section 1.5. See also Section 2-2 of **K**.) The first of these is the linear transformation we have been calling D^2, and the second is the identity transformation; thus it is consistent with our previous notation to write

$$L = D^2 + 1$$

This gives a very convenient notation for linear differential operators; for example, the transformation

$$f \longmapsto 3f''' - f' + f$$

(with domain $\mathcal{C}^3(I)$ for some interval I) is denoted by $3D^3 - D + 1$. Differential operators of this type are important because differential equations of the form

$$Af = g,$$

with A an operator of this type, arise quite frequently in applications and can always be solved in principle. In the following reading passage this class of operators, the so-called *constant-coefficient linear differential operators*, is defined formally.

READ Section 2-3 on pages K48–K53 *omitting Example 5.*

Much of this reading passage is revision of work we studied previously in N.

Notes

(i) *line 5, et seq. page* K51. We shall not be interested in the concept of *nilpotence* here.

(ii) *line −8, page* K51. "... factorization of polynomials depends only ..." This bit is not strictly true. If you examine in detail all the properties of ordinary arithmetic that you use in order to write

$$(x+2)(x+3)(x-1) = x^3 + 4x^2 + x - 6,$$

you will find that there are others than those mentioned by **K**. Fortunately, they are all properties which linear transformations possess.

(iii) *line −6, page* K51. Examples of functions in $\mathcal{C}^\infty[a, b]$ are

$$x \longmapsto 1 \qquad (x \in [a, b])$$
$$x \longmapsto \exp x \qquad (x \in [a, b])$$

(iv) *line 5, page* K52. **K** does not distinguish between functions and images. In the notation of the Foundation Course, which does make this distinction, we would either regard y as a function throughout, and write

$$(D^2 + D - 2)y = y'' + y' - 2y$$

or else regard it as a symbol for the image of x under a function denoted by some other letter, say Y, in which case the equation would have to be written

$$((D^2 + D - 2)Y)(x) = Y''(x) + Y'(x) - 2Y(x)$$

or

$$((D^2 + D - 2)Y)(x) = \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y$$

When we come to manipulate specific functions, the notation used in **K** has its advantages. The authors' explanation of their notation is given at the end of this reading passage.

Exercise

Exercise 7, page K54; parts (a), (b), (c) and (d).

Solution

The answers are given on page K728. The method is a straightforward application of the definition; e.g. (using the **K** notation for functions)

$$(D^2 - 1)2e^x = D^2(2e^x) - 1 \cdot 2e^x = 2e^x - 2e^x = 0$$

For (c) and (d) there are two alternative methods either using the bracketed operators in succession, or multiplying their product out first

$$\begin{aligned}(D + 1)(D - 2)e^x &= (D + 1)(De^x - 2e^x) = (D + 1)(-e^x)\\ &= -e^x - e^x = -2e^x\end{aligned}$$

or

$$\begin{aligned}(D + 1)(D - 2)e^x &= (D^2 - D - 2)e^x\\ &= (D^2e^x - De^x - 2e^x)\\ &= e^x - e^x - 2e^x = -2e^x\end{aligned}$$

4.1.4 Linear Differential Operators in General

To be able to deal with as wide a class of linear differential equations as possible, it is useful to consider linear differential operators more general than the constant-coefficient type defined in the previous sub-section. This generalization is given in the next reading passage.

READ Section 3-1 from page K86 *as far as* "... the operator itself." *on line 6 of page* K88.

Notes

(i) *line 4, page* K86. By $xD^2 + D + x$ is meant the operator which maps any real function $f \in \mathcal{C}^2(I)$ to the real function

$$x \longmapsto xf''(x) + f'(x) + xf(x) \qquad (x \in I)$$

(See also Example 5, page K52.)
(ii) *Definition (3-1), page* K86. As an example, $xD^2 + D + x$ is a linear differential operator, of order 2. The choice of $\mathcal{C}^n(I)$ for the domain ensures that D, $D^2, \ldots, D^n$ all produce continuous functions, and multiplying these continuous functions by the continuous functions $a_0, a_1, \ldots, a_n$, gives further continuous functions, so that all the images under L, obtained by adding these continuous functions, are themselves continuous. However the operator $y \longmapsto xy'' + yy' + y^{1/2}$ cannot be expressed in the form of Equation (3-1) and is not linear.
(iii) *line −4, page* K86. "$a_n(x)$ is not identically zero on I" means "$a_n(x)$ is not equal to zero for all $x \in I$" or in other words "a_n is not the zero function on I".
(iv) *lines 1 and 3, page* K88. The notation used in K for intervals is defined in the footnotes on pages K3 and K88. Here is a full specification.

For $a, b \in R$, we have

$[a, b] = \{x : a \leqslant x \leqslant b\}$: a and b both lie in the interval
$(a, b) = \{x : a < x < b\}$: neither a nor b lies in the interval
$[a, b) = \{x : a \leqslant x < b\}$: a, but not b, lies in the interval
$(a, b] = \{x : a < x \leqslant b\}$: b, but not a, lies in the interval.

These intervals are usually referred to as being "closed", "open" and "half-open" respectively. Unfortunately the symbol (a, b) has two meanings: the interval defined above, and the ordered pair whose two elements are a and b. (That is why we used the notation $]a, b[$ for open intervals in the Foundation Course.)

Where relevant, we write $+\infty$ ("plus infinity") for b and $-\infty$ for a. This is just a notational trick; there are not points actually called $+\infty$ or $-\infty$ in R. Thus, when the symbols ∞ and $-\infty$ are used, they go with a round bracket rather than a square bracket, to indicate that they are not themselves included in the intervals they define. We have the following possibilities:

$$[a, +\infty) = \{x : a \leqslant x\}$$
$$(a, +\infty) = \{x : a < x\}$$
$$(-\infty, b] = \{x : x \leqslant b\}$$
$$(-\infty, b) = \{x : x < b\}$$
$$(-\infty, +\infty) = R$$

(v) *line 2, page* K88. The coefficients of D^2, D, etc., can be any continuous functions. All that is required for linearity is that the operator is equivalent to the form of Equation (3-1). The notation ln denotes the natural logarithm function.

Exercises

1. Exercise 1(c) and 1(d) on page K88. If you require further practice try Exercises 2, 3 on K88, 89.
2. State the order of the linear differential operators in Exercise 2 page K88, on the interval (0, 1].

Solutions

1. The answers are given on page K731.
   In (d) you must apply the bracketed operators one at a time to the function. The fact that

$$(D + 1)(D - x) \neq D^2 + (1 - x)D - x$$

   is taken up in the next sub-section.
2. (a) 2 if $a \neq 0$; 1 if $a = 0$, $b \neq 0$; 0 if $a = b = 0$.
   (b) 2. (c) 2.

4.1.5 Products of Linear Differential Operators

We saw earlier (top of page K52) that constant-coefficient linear differential operators can be multiplied just like ordinary polynomials, so that, for example

$$(D + 2)(D - 1) = (D - 1)(D + 2) = D^2 + D - 2.$$

For general linear operators, however, multiplication does not follow this simple law, and in fact it is not even commutative. An example illustrating the procedure for multiplying such operators is given in the next reading passage.

READ the remainder of Section 3-1 on page K88.

It is crucial to remember that an operator L is not a disembodied expression but has a meaning according to its effect on a typical element of its domain. Whenever in doubt about re-expressing the form of L, write out $L(y)$ in full.

Exercises

1. Verify the non-commutativity of the multiplication of two general linear differential operators by carrying out calculations similar to the one in the reading passage, for:
   (i) the product $(xD + 2)(3D + 1)$ and
   (ii) the product of the same two operators in the reverse order.
2. Exercise 4, parts (a), (b), (c) on page K89.

Solutions

1. (i) A calculation similar to the one on page K88 gives

$$\begin{aligned}(xD + 2)(3D + 1)y &= (xD + 2)(3y' + y)\\ &= 3xy'' + xy' + 6y' + 2y\\ &= (3xD^2 + (x + 6)D + 2)y\end{aligned}$$

   (ii) On the other hand

$$\begin{aligned}(3D + 1)(xD + 2)y &= (3D + 1)(xy' + 2y)\\ &= 3xy'' + 3y' + 6y'\\ &\quad + xy' + 2y\\ &= (3xD^2 + (x + 9)D + 2)y\end{aligned}$$

   thus $(xD + 2)(3D + 1) = 3xD^2 + (x + 6)D + 2$
   but $(3D + 1)(xD + 2) = 3xD^2 + (x + 9)D + 2$

2. (a) $(D^2 + 1)(D - 1) = D^3 - D^2 + D - 1$
   Being constant-coefficient linear differential operators, $(D^2 + 1)$ and $(D - 1)$ can be multiplied like ordinary polynomials.

(b) Following the method on page K88, we apply the differential operators to an arbitrary function y in an appropriate vector space ($\mathcal{C}^2(I)$ will do here, though it is not in general necessary to specify this).

$$\begin{aligned}(xD(D-x))y &= xD(y'-xy)\\ &= xy'' - x^2y' - xy\end{aligned}$$

i.e. $\quad xD(D-x) = xD^2 - x^2D - x$

(c) $$\begin{aligned}(xD^2+D)^2f &= (xD^2+D)(xf''+f')\\ &= xD(xf^{(3)}+f''+f'') + (xf^{(3)}+f''+f'')\\ &= x(xf^{(4)}+f^{(3)}+2f^{(3)}) + (xf^{(3)}+2f'')\\ &= x(xf^{(4)}+4f^{(3)})+2f''\end{aligned}$$

Therefore $(xD^2+D)^2f = x^2D^4 + 4xD^3 + 2D^2$.

In writing out operators with non-constant coefficients, it is essential to write the coefficients (x, $-x^2$, $-x$, in part (b) above) to the *left* of the D symbols (just as in Equation (3-1) on page K86). This is to keep them out of the line of fire of the D operator, which acts on the functions written to the *right* of it.

4.1.6 Summary of Section 4.1

In this section we defined the terms

continuously differentiable	(page K13)	* * *
interval	(page K3)	* * *
linear differential operator	(page K86)	* * *
order	(page K86)	* * *

We introduced the notation

$\mathcal{C}(I)$	(page K86)
$\mathcal{C}[a, b]$	(page K3)
D^2, D^n	(page K50)
$\mathcal{C}^n(I)$	(page K86)
(a, b)	(page C16)
(a, ∞)	(page C16)
$(-\infty, b)$	(page C16)
$(-\infty, \infty)$	(page C16)

Techniques

1. Use $(aD+b)(cD+d) = acD^2 + (ad+bc)D + bd$ when a, b, c and d are constant functions. * * *
2. When a, b, c and d are not constant functions, apply the operator in 1 by applying each factor in turn. * * *

4.2 LINEAR DIFFERENTIAL EQUATIONS

4.2.0 Introduction

You have already met some linear differential equations in the Foundation Course; for example the equation

$$f'' + f = 0 \qquad (f: R_0{}^+ \longrightarrow R)$$

mentioned earlier in this unit, which models the vibration of a mass-and-spring system. Using the notation we have developed in this unit the equation can be written in terms of the linear differential operator $D^2 + 1$, in the form

$$(D^2 + 1)f = 0$$

where $f \in \mathcal{C}^2(R_0{}^+)$.

Thus the problem of solving the linear differential equation is equivalent to finding the kernel of the linear transformation $D^2 + 1$ (the inverse image of the element 0 in the codomain). Such an equation is called a *linear problem* by N (page N63); K's terminology is *operator equation* (page K80).

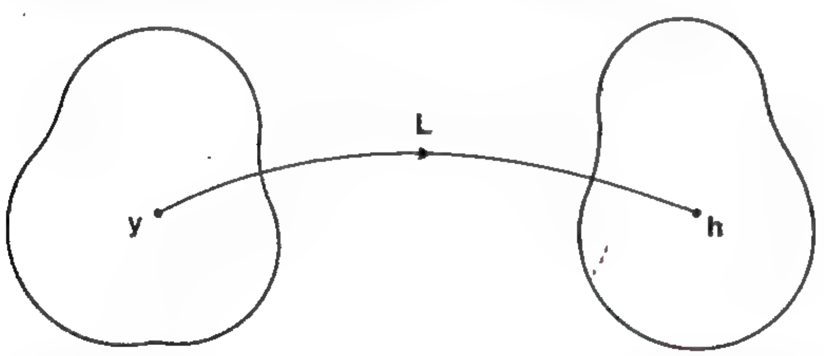
This point of view makes it possible to exploit the general theory of linear transformations in solving such differential equations. You have already met all the main ideas of this section in the Foundation Course (*Units M100 24* and *31*, *Differential Equations I* and *II*); all we do here is to revise these ideas, define the technical terms more precisely, and extend them to the general linear differential equation.

4.2.1 Definitions

READ page K91 *of Section 3-2.*

Notes

(i) *line 8* (*equation* (*3-5*)), *page* K91. This equation can also be written $L(y) = h$, a form which emphasizes the fact that both y and h are *functions* and that L is a linear transformation from one function space to another.



(ii) *line 10.* "defined on I" means that the domain of L is $\mathcal{C}^n(I)$.

"h is identically zero on I" means that $h(x) = 0$ for all x in I, or in other words h is the zero function with domain I.

(iii) *lines 11–12. Normal* means that the image of I under the function a_n does not include 0; i.e. $a_n(x) \neq 0$, for all $x \in I$. The word "vanishes" is often used to mean "is equal to 0".

(iv) *line 13.* The reason for requiring $y \in \mathcal{C}^n(I)$ is to be sure that it is in the domain of L. "Satisfies the equation identically on I" means that $Ly(x) = h(x)$ holds for all $x \in I$, or, in other words, that the functions y and h (with domain I) satisfy the equation $L(y) = h$.

(v) *line* -1. "analysis" means the branch of mathematics that derives from the concept of a limit; it includes, for example, the differential and integral calculus (treated rigorously) and the theory of differential equations.

Exercises

1. For each of the following differential equations, state whether or not it is linear.

   (i) $5x^2y'' + 2y = 7x^{2/3}$
   (ii) $(e^x y)'' = 5xy$
   (iii) $(y'')^2 + 5x^2y = 7$
   (iv) $y' + 5xy^2 = 0$
   (v) $\dfrac{y'}{y} = 4x$ (given that $y(x) \neq 0$ for all x).
   (vi) The population growth equation $y' = ay$ where a is some real number.

2. For each of the following linear differential equations, state whether or not it is (a) homogeneous, (b) normal.

   (i) $(2e^xD^2 + xD + e^x)y = 0 \qquad (x \in [-1, 1])$
   (ii) $(2\cos xD^2 + D)y = 5x \qquad (x \in [-1, 1])$
   (iii) $(2\cos xD^2 + D)y = 0 \qquad (x \in [-2, 2])$
   (iv) $(x - |x|)y'' + y' = x - |x| \qquad (x \in [-1, 1])$
   (v) $(x - |x|)y'' + y' = x - |x| \qquad (x \in [1, 2])$

3. Which of the following are solutions of the differential equation

$$(D^2 - 4)y = 4e^{2x} \qquad (x \in R)?$$

   (i) $y = e^{2x}$
   (ii) $y = xe^{2x}$
   (iii) $y = (1 + x)e^{2x}$

Solutions

1. (i) Linear: it can be written $(5x^2D^2 + 2)y = 7x^{2/3}$.
In terms of the Equations (3-5) and (3-6) on page K91,

$$\begin{aligned} L &= 5x^2D^2 + 2 \\ h(x) &= 7x^{2/3} \\ n &= 2 \\ a_2(x) &= 5x^2 \\ a_1(x) &= 0 \\ a_0(x) &= 2. \end{aligned}$$

(ii) Linear: it can be written $(e^xD^2 + 2e^xD + e^x - 5x)y = 0$.
(iii) Not linear: the term $(y'')^2$ spoils it.
(iv) Not linear: the y^2 spoils it.
(v) Linear: as $y(x)$ is never zero, it can be written as $y' = 4xy$, i.e. $(D - 4x)y = 0$.
(vi) Linear: it can be written $(D - a)y = 0$.

2. (i) (a) Homogeneous, because the $h(x)$ of Equation (3-5) is zero.
(b) Normal: $2e^x \neq 0$, for all $x \in [-1, 1]$.
(ii) (a) Not homogeneous: $h(x) = 5x$.
(b) Normal
(iii) (a) Homogeneous
(b) Not normal: $\cos x = 0$ when $x = \pm\dfrac{\pi}{2}$.
(iv) (a) Not homogeneous: $x - |x| \neq 0$, $x < 0$.
(b) Not normal: $x - |x| = 0$, $x \geqslant 0$.
(v) (a) Homogeneous
(b) Normal: since $x - |x|$ is zero *throughout* the domain, rather than merely at certain points in it,

the equation reduces in this domain to a *first-order* normal linear differential equation, namely

$$y'(x) = 0 \qquad (x \in [1, 2])$$

3. (i) NO: $(D^2 - 4)e^{2x} = 4e^{2x} - 4e^{2x}$
$= 0$
$\neq 4e^{2x}$.

(ii) YES: $(D^2 - 4)xe^{2x} = D(e^{2x} + 2xe^{2x}) - 4xe^{2x}$
$= 2e^{2x} + 2e^{2x} + 4xe^{2x} - 4xe^{2x}$
$= 4e^{2x}$.

(iii) YES: $(D^2 - 4)(1 + x)e^{2x} = (D^2 - 4)e^{2x}$
$+ (D^2 - 4)xe^{2x}$
by linearity,
$= 0 + 4e^{2x}$
$= 4e^{2x}$.

4.2.2 Homogeneous Equations

As a start in exploiting the linear algebra covered in earlier units to solve differential equations, we consider the special case in which $h = 0$.

READ the first paragraph on page K92 *(down to* "... choice of terminology."*), followed by Example 1 on pages* K92–3.

Notes

(i) *line 5, page* K92. "of the x-axis"–this phrase merely indicates that the variable x will be used later on for elements in the domain.
(ii) *line 6, page* K92. "Null space" is the term used in K for what we have been calling the kernel of a linear transformation: the inverse image of the zero vector in the codomain. We discussed the kernel in the Foundation Course and also in *Unit 2, Linear Transformations* (see page N31); the discussion in K (with some new examples) begins on page K55, but you need not study it unless you wish to revise and have time to do so.
(iii) *line 5, page* K93. The solution set defined by Equation (3-11) can also be written $\langle \sin x, \cos x \rangle$ (see page N12).

Exercise

(i) How many vectors are needed to span the solution space of $(D^2 - 4)y = 0$? (Domain of y is R.)
(ii) Show that the functions $y = e^{2x}$ and $y = e^{-2x}$ (with domain R) are solutions of the differential equation in (i). Are they linearly independent?
(iii) Write down the general solution of the differential equation in (i).

Solution

(i) Since the equation is normal, homogeneous and has order 2 (the operator $D^2 - 4$ contains D^2 but no higher power of D), the solution space has dimension 2.
(ii) They are solutions, because $(D^2 - 4)e^{2x} = 0$ and $(D^2 - 4)e^{-2x} = 0$. They are also linearly independent. To prove this we show that

$$c_1 e^{2x} + c_2 e^{-2x} = 0 \qquad (x \in R)$$

implies $c_1 = c_2 = 0$ (compare the first 3 lines of page K93). If, for example, we set $x = 0$ and $x = 1$ in the equation, then

$$c_1 + c_2 = 0$$
$$c_1 e^2 + c_2 e^{-2} = 0$$

which, solving for c_1 and c_2, give $c_1 = c_2 = 0$.

(iii) Since e^{2x} and e^{-2x} are linearly independent and the solution space has dimension 2, these functions form a basis. Thus the solution set is $\langle e^{2x}, e^{-2x}\rangle$ and the general solution is $y = c_1e^{2x} + c_2e^{-2x}$, where c_1 and c_2 are arbitrary.

4.2.3 Nonhomogeneous* Equations

We saw in *Unit 3, Hermite Normal Form* (page N63) and also in various places in the Foundation Course that the solution set of any linear problem

$$Ly = h$$

is given by

$$\text{solution set} = \{y_0\} + \{\text{kernel of } L\};$$

where y_0 is any solution of the given linear problem, and the notation means that the solution set consists of all vectors (in the domain of L) having the form

$$y_0 + \text{an element of the kernel.}$$

The next reading passage explains how we can exploit this particularly simple structure of the solution set.

READ the remaining paragraph on page K92 *and Example 2 on page* K93.

Example

Find the particular solution of

$$y'' + y = x$$

such that $y(0) = 2$, $y'(0) = 0$.

From the reading passage, any solution has the form

$$y(x) = x + c_1 \sin x + c_2 \cos x$$
$$c_1, c_2 \text{ arbitrary.}$$

So that

$$y'(x) = 1 + c_1 \cos x - c_2 \sin x$$

The two given conditions determine c_1 and c_2. Thus

$$y(0) = c_2 = 2$$
$$y'(0) = 1 + c_1 = 0 \therefore c_1 = -1.$$

Thus the particular solution we require is

$$y(x) = x - \sin x + 2 \cos x.$$

Exercises

1. Write down the general solution of the differential equation $(D^2 - 4)y = 4e^{2x}$. You will find useful information in the answers to Exercise 3 of sub-section 4.2.1 and the Exercise of sub-section 4.2.2.
2. Find the particular solution of $(D^2 - 4)y = 4e^{2x}$ with the properties $y(0) = 1$, $y'(0) = 3$.

Solutions

1. By Exercise 3 of sub-section 4.2.1, the equation $(D^2 - 4)y = 4e^{2x}$ has $y = xe^{2x}$ and $y = (1 + x)e^{2x}$ as particular solutions, and by the Exercise of sub-section 4.2.2, the associated homogeneous equation $(D^2 - 4)y = 0$ has $y = c_1e^{2x} + c_2e^{-2x}$ as its

* We follow K in dropping the hyphen.

general solution. The general solution of $(D^2 - 4)y = 4e^{2x}$ is therefore

$$y = xe^{2x} + c_1e^{2x} + c_2e^{-2x}$$

or

$$y = (1 + x)e^{2x} + c_1e^{2x} + c_2e^{-2x}.$$

The two general solutions are equivalent: the second can be written

$$y = xe^{2x} + (1 + c_1)e^{2x} + c_2e^{-2x}$$

which has the same form as the first since it does not matter whether we call the arbitrary constant c_1 or $1 + c_1$.

2. We want to choose the constants c_1, c_2 in the general solution

$$y(x) = xe^{2x} + c_1e^{2x} + c_2e^{-2x} \qquad (x \in R)$$

to satisfy

$$y(0) = 1, y'(0) = 3.$$

Now

$$y(0) = 0 + c_1 + c_2$$

so that

$$c_1 + c_2 = 1$$

Also

$$y'(x) = (1 + 2x)e^{2x} + 2c_1e^{2x} - 2c_2e^{-2x}$$

therefore

$$y'(0) = 1 + 2c_1 - 2c_2$$

i.e.

$$c_1 - c_2 = 1,$$

from which $c_1 = 1$ and $c_2 = 0$; so that the required solution is $y(x) = (1 + x)e^{2x}$

If you feel you need more practice on this type of problem, try any of Exercises 3, 4 and 7 on page K95. (In Exercise (7a), $\sinh x$ means $\frac{1}{2}(e^x - e^{-x})$ and $\cosh x$ means $\frac{1}{2}(e^x + e^{-x})$.)

4.2.4 Nonlinear Equations (Optional)

The discussion from "Before leaving this section. · · " on page K93 to the end of the section on page K94 shows what goes wrong if we try to apply the theory to a nonlinear equation. Similar equations were discussed in the Foundation Course (*Unit M100 24, Differential Equations I*). They are not essential to this course, so you need not study the passage.

4.2.5 Summary of Section 4.2

In this section we defined the terms

linear differential equation	(page K91)	* * *
homogeneous*	(page K91)	* * *
nonhomogeneous*	(page K91)	* * *
normal	(page K91)	* * *
solution*	(page K91)	* * *
null space (kernel)	(page K55)	* * *
general solution*	(page K92)	* * *
particular solution*	(page K92)	* * *

Theorems

1. (Page K92)
   The dimension of the null space of L, a linear differential operator, equals the order of L. * * *

2. (Page K92)
   The solution set of a linear differential equation $Ly = h$ is * * *

   $$\{y: Ly = h\} = \{y_p\} + \{y: Ly = 0\}$$

   where $Ly_p = h$.

* These terms we introduced in *Unit 3, Hermite Normal Form.*

4.3 FIRST-ORDER LINEAR EQUATIONS

4.3.0 Introduction

The rest of this unit deals with the first-order linear differential equations, one of the simplest types of linear differential equation. The main ideas have already been discussed in the Foundation Course (*Unit M100 24, Differential Equations I*); in this section we will revise these ideas and also learn about some of the manipulations that are necessary to solve such equations.

4.3.1 Homogeneous Equations

As in the general theory studied in Section 2, we begin with homogeneous equations of the form

$$Ly = 0$$

in which we restrict L to be first order; that is, it has the form

$$L = a_1(x)D + a_0(x)$$

If the equation is normal, then since it is first order, its solution space is one-dimensional and so it has a basis consisting of a single function; we can solve the equation completely if we can find such a basis, i.e. a single solution. The first reading passage shows how to find such a solution, using the method of *separation of variables* which you met in the Foundation Course (*Unit M100 24, Differential Equations I*), and how to obtain the general solution from it.

READ Section 3-3 from page K95 *to* "... where c is an arbitrary constant" *in the middle of page* K96.

Notes

(i) *line 6, page* K96. Why do we need $a_1(x) \neq 0$? Can we be sure that this condition is satisfied? If $a_1(x) = 0$ somewhere in I, then we would be dividing by zero for that value of x in the rearrangement of Equation (3-17); we avoid this possibility by only considering normal equations, which are just those equations with leading coefficient $a_1(x)$ non-zero *everywhere* in the interval I.

(ii) *line 7, page* K96. When writing the symbol $\frac{1}{y}$ we make the assumption that $y(x) \neq 0$ for all x in I. Consequently the separation of variables will not yield any solution that violates this assumption, in particular the solution $y = 0$ (the zero function). We shall, however, recover any solution or solutions left out at this stage, when we construct the general solution at the end of the calculation: at this stage all we need is any one solution.

(iii) *line 9, page* K96. In the notation of the Foundation Course, the integral on the right-hand side of this equation would be written

$$\int \frac{a_0}{a_1} \quad \text{or} \quad \int x \longmapsto \frac{a_0(x)}{a_1(x)}$$

or, in "abused" notation

$$\int \frac{a_0(x)}{a_1(x)}.$$

The notation used in **K** is the Liebniz notation. An introduction to Leibniz notation was given in the Foundation Course (*Unit M100 13, Integration II*), and a summary of it is given in the supplementary handbook **TI***.

(iv) *line 9, page* K96. On the left-hand side of the equation, the integral $\int \frac{1}{y}\frac{dy}{dx}\,dx$ has been evaluated using integration by substitution. The principle behind this technique was explained in the Foundation Course unit referred to

* TI stands for *Techniques of Integration*.

above. Here all we need is the formula using the technique, which is reproduced in paragraph 6 of the section *Rules of Integration* of TI. The formula is

$$\int f(y(x))y'(x)\,dx = \int f(u)\,du$$

where u stands for $y(x)$. Taking f to be the reciprocal function $x \longmapsto \frac{1}{x}$, we get

$$\int \frac{1}{y(x)}\,y'(x)\,dx = \int \frac{1}{u}\,du \quad \begin{Bmatrix} x \in I \\ u \in y(I) \end{Bmatrix}$$

The integral at the right is a standard form: since

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

we have

$$\int \frac{1}{u}\,du = \ln u \qquad (u \in \text{suitable domain})$$

But what is the suitable domain? Since $\ln u$ is defined only for positive u, we can take the domain to be at most R^+. Thus we obtain, putting back $y(x)$ for u

$$\int \frac{1}{y(x)}\,y'(x)\,dx = \ln y(x) \qquad (y(x) \in R^+).$$

If we assume, therefore, that $y(x)$ is positive for all $x \in I$, then the left-hand side of the equation on line 9 can be taken as $\ln y$. We suggest that you amend the left-hand side of the equation to $\ln y$ and that of the next to y. The modulus signs are not necessary (they are put in to allow for solutions where $y(x) < 0$ for all $x \in I$, but it is not necessary to consider them) since a positive solution does exist and we only need one solution. It is important to realize that we need just *one* solution at this stage: see next note.

(v) *line 11, page* K96. The theorem referred to is: the solution space of a normal homogeneous linear differential equation of order n has dimension n.

Here n is 1 and so the single solution we have found is a basis for the entire solution space; this space is in fact

$$\left\langle \exp\left(-\int \frac{a_0(x)}{a_1(x)}\,dx\right)\right\rangle.$$

The general description is often difficult to follow, so we give an example of a particular case.

Example

Find the general solution of

$$y' + ay = 0,$$

where $a \in R$ and the domain of y is R. ($y \in C^1(R)$.)

This is of the form of Equation (3-17) on page K96 with $a_1(x) = 1$, $a_0(x) = a$, and so the solution is given by the formula at the end of this reading passage. The integral in that formula yields

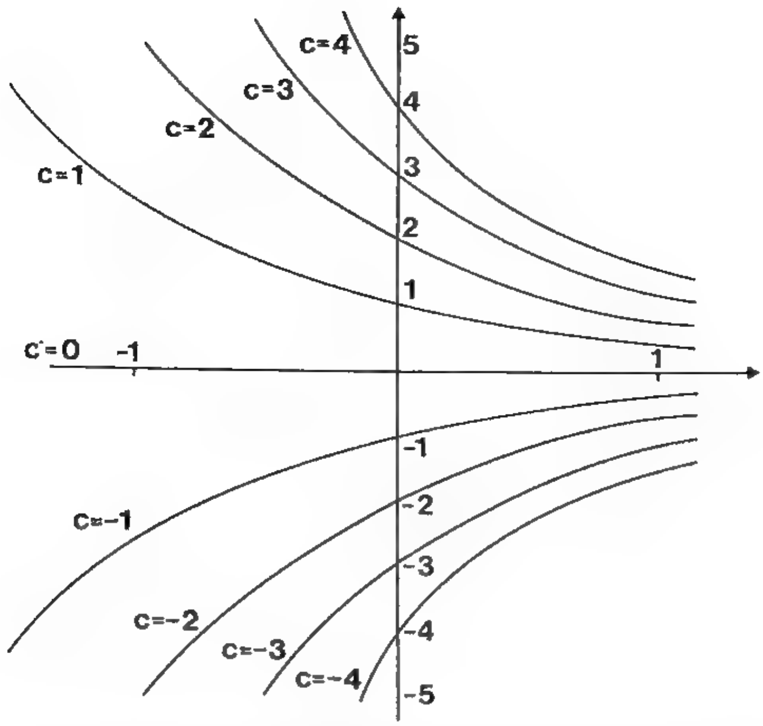
$$\int \left(\frac{a}{1}\right) dx = a\int dx = ax.$$

(We choose zero as the constant of integration here, because we only need one solution of the equation as a basis for the solution space. The arbitrary constant will go in later when we express the entire solution space in terms of this basis.)

Substituting the integral we have just evaluated into the formula at the end of the reading passage, we obtain

$$y(x) = ce^{-ax}$$

as the general solution; i.e. the solution set is $\langle e^{-ax}\rangle$. For $a = 1$, the solution set is illustrated in the diagram.



At the end of any calculation, such as the one we have just done to solve the equation $y' + ay = 0$, it is advisable to check the result. Here we have

$$y'(x) = -ace^{-ax}$$
$$ay(x) = ace^{-ax}$$

so that $y'(x) + ay(x) = 0$ and the result is verified. We suggest that you work through this example following the steps from Equation (3-17) to the end explicitly.

Exercises

1. Find $\int xe^{x^2}\,dx$, by using the substitution $u = x^2$.
2. Exercise 1, page K99. Take the domain of y to be R^+.
3. Exercise 3, page K99. Take the domain of y to be $(0, \frac{1}{2}\pi]$, and use the substitution $u = \sin x$.

Solutions

1. The formula for the substitution is

$$\int f(y(x))y'(x)\,dx = \int f(u)\,du$$

where $u = y(x)$ which, in this case, is x^2. Therefore

$$y'(x) = 2x.$$

If we now take $f = \exp$, then we have

$$\int \exp(x^2)\cdot 2x\,dx = \int \exp u\,du$$
$$= \exp u.$$

The only difference between this and the integral we want to evaluate is the 2 in the integral on the left-hand side; so we conclude

$$\int x \exp(x^2)\,dx = \tfrac{1}{2}\exp u = \tfrac{1}{2}\exp x^2.$$

It is worth checking the answer by differentiation.

2. $$xy'(x) + 2y(x) = 0 \qquad (x \in R^+)$$

Here $a_1(x) = x$, $a_0(x) = 2$, using the notation of the reading passage. The integral we need is

$$\int \frac{a_0(x)}{a_1(x)}\,dx = 2\int \frac{dx}{x} = 2 \ln x$$

since $x \in R^+$.

The general solution is therefore

$$y(x) = ce^{-2\ln x} = \frac{c}{x^2} \qquad (x \in R^+)$$

(Remember exp and ln are inverse functions; ln has R^+ for domain.)

3. $$(\sin x)y'(x) + (\cos x)y(x) = 0 \qquad (x \in (0, \tfrac{1}{2}\pi])$$

The integral we need is

$$\int \frac{a_0(x)}{a_1(x)}\,dx = \int \frac{\cos x}{\sin x}\,dx \qquad (x \in (0, \tfrac{1}{2}\pi]).$$

The suggested substitution is $u = \sin x$. To use this we apply the substitution formula. The formula becomes

$$\int f(\sin x) \cos x\,dx = \int f(u)\,du$$

and with f the reciprocal function $x \longmapsto \dfrac{1}{x}$, we find that

$$\int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{u}\,du = \ln u = \ln(\sin x),$$

since $\sin x$ is positive for all x in the domain $(0, \tfrac{1}{2}\pi]$. Thus the general solution is

$$y(x) = ce^{-\ln(\sin x)} = \frac{c}{\sin x} \qquad (x \in (0, \tfrac{1}{2}\pi])$$

4.3.2 Nonhomogeneous Equations

Now that we know how to solve the general homogeneous equation of first order (that is, how to reduce it to the evaluation of an integral), we can also solve any nonhomogeneous equation for which we can find a particular solution. The next reading passage describes a general method for obtaining the necessary particular solution.

READ from "To obtain a particular solution . . ." *on page* K96 *as far as, but not including, Example 3 on page* K98.

Notes

(i) *line −8, page* K96. The basic idea of the method is the same as for the homogeneous equation: we multiply both sides of the equation by a factor which makes it possible to integrate both sides *even though we do not know the function* y. Such a factor is called an *integrating factor*. The factor $1/[a_1(x)y(x)]$ which worked for the homogeneous equation no longer works now, because we cannot integrate $h(x)/[a_1(x)y(x)]$ without knowing y. Fortunately, even for the nonhomogeneous equation there is a fairly simple integrating factor,

$$\frac{1}{a_1(x)} \exp \int \frac{a_0(x)}{a_1(x)}\, dx,$$

for the equation we are studying. In K this integrating factor appears from nowhere, but there is a plausible argument for obtaining it; since this argument is given in *Unit M100 24, Differential Equations I*, we shall not treat it in this course.

(ii) *line 5, page* K97. This is an important point: it is much better to understand how a method works and be able to work out or look up the formulas when they are needed, than to try to memorize complicated expressions like Equation (3-20). Here the main thing to remember is that an integrating factor for an equation of the form

$$y'(x) + P(x)y(x) = Q(x)$$

is $\exp \int P(x)\, dx$. In using the procedure you must always remember to divide through by $a_1(x)$ first. This is always possible if the equation is normal. It is not even necessary to solve the homogeneous equation first, as the examples that follow show.

(iii) *line −11, page* K97. Notice the difference from the general method described at the beginning of the reading passage. Instead of completely solving the associated homogeneous equation first and then finding a particular solution of the nonhomogeneous equation separately, we do both parts at once by including a constant of integration c when we do the main integration (although we choose zero as the constant of integration in the integration that gives the integrating factor—e^{x^2} here—because one integrating factor will do to solve the equation).

(iv) *line −9, page* K97. For $\int xe^{x^2}$, see Exercise 1 of sub-section 4.3.1.

(v) *line −5, page* K97. The domain of y does not *have* to be $(0, \infty)$ or $(-\infty, 0)$; it could be any interval that is a subset of one of these.

(vi) *line −1, page* K97. The use of $|x|$ where you might expect x is a trick for solving two separate problems with the same calculation: one problem referring to the interval $(0, \infty)$ (i.e. when $x > 0$) and one to the interval $(-\infty, 0)$. (See Section III.2.1 of TI.)

(vii) *line 11, page* K98. "defined on the entire real line" means "with domain R".

(viii) *line 13, page* K98. "potentially misleading". Note the warning. It is often convenient to forget about the interval on which the functions considered are defined and whether or not the functions are continuous. This is all very well until something goes wrong, but even if nothing does apparently go wrong, we might have an error and we cannot be absolutely sure of our answer unless we have applied the theory correctly, that is if we have used the theory only in those situations where we know it can be applied.

You may have realized that all the examples in this section are carefully contrived to give simple results. In real life, things are not so easy. It

may not be possible, for example to do the integrations; thus, in Example 1 on page K97, if the right-hand side of the differential equation had been 1 instead of x, we would have had the integral $\int e^{x^2}\,dx$ (in place of $\int xe^{x^2}\,dx$), and this integral cannot be expressed in terms of elementary functions.* Another thing that can give trouble is that, even when the integral can be done, the formula it leads to may be too complicated to be useful. In either case, it may be necessary to find some other method (perhaps a numerical method) for solving the equation. We shall be looking at some of these numerical methods later in the course. An obvious modification, which will help in some cases, is still to use the method described as far as the integral form of the solution in Equation (3-20), and then to use a numerical method, such as Simpson's rule, to evaluate the integrals over a specified interval.

Exercises

1. Find the general solution of

$$2y'(x) + y(x) = e^x \qquad (x \in R).$$

2. Find the solution of

$$2y'(x) + y(x) = x \qquad (x \in R)$$

that satisfies $y(0) = 0$.

Solutions

1. $$2y'(x) + y(x) = e^x \qquad (x \in R)$$

Divide by $a_1(x)$:

$$y'(x) + \frac{1}{2}y(x) = \frac{1}{2}e^x.$$

An integrating factor is

$$\exp\int\frac{1}{2}\,dx = \exp\left(\frac{1}{2}x\right).$$

The equation becomes

$$y'(x)e^{(1/2)x} + \frac{1}{2}y(x)e^{(1/2)x} = \frac{1}{2}e^{(3/2)x}$$

i.e.

$$\frac{d}{dx}(y(x)e^{(1/2)x}) = \frac{1}{2}e^{(3/2)x}.$$

Integration gives

$$y(x)e^{(1/2)x} = \frac{1}{2}\int e^{(3/2)x}\,dx + c$$

i.e.

$$y(x)e^{(1/2)x} = \frac{1}{3}e^{(3/2)x} + c.$$

Hence the solution is

$$y(x) = \frac{1}{3}e^x + ce^{-(1/2)x} \qquad (x \in R).$$

* The elementary functions are x^n, $\ln x$, $\exp x$, $\sin x$, $\cos x$, etc.

2. : $2y'(x) + y(x) = x$

Since the linear operator, $2D + 1$, is the same as in Exercise 1, the integrating factor and the kernel are also the same as there. The method of Solution 1 gives

$$\frac{d}{dx}(y(x)e^{(1/2)x}) = \left(y'(x) + \frac{1}{2}y(x)\right)e^{(1/2)x}$$
$$= \frac{1}{2}xe^{1/2x}$$

so that

$$y(x)e^{(1/2)x} = \frac{1}{2}\int xe^{(1/2)x}\,dx + c.$$

To integrate $\frac{1}{2}xe^{(1/2)x}$, we use integration by parts.

$$\int f \times Dg = f \times g - \int g \times Df$$

or in Leibniz notation

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Let

$$f(x) = x$$
$$g(x) = e^{(1/2)x}.$$

Formula yields

$$\int x\frac{d}{dx}(e^{(1/2)x})\,dx = xe^{(1/2)x} - \int e^{(1/2)x}\,dx.$$

Therefore

$$\int \frac{1}{2}xe^{(1/2)x}\,dx = xe^{(1/2)x} - 2e^{(1/2)x} = (x-2)e^{(1/2)x}.$$

Thus

$$y(x)e^{(1/2)x} = (x-2)e^{(1/2)x} + c$$

and the general solution is

$$y(x) = x - 2 + ce^{-(1/2)x}.$$

We want $y(0) = 0$,

i.e.

$$0 = -2 + c.$$

So the required solution is

$$y(x) = x - 2 + 2e^{-(1/2)x} \qquad (x \in R).$$

If you require further practice, try Exercises 5 and 7 on page K99.

4.3.3 Bernoulli's Equation

The last reading passage in Section 3–3 of K deals with a special type of non-linear first-order differential equation which can be solved by means of a substitution converting it to a linear equation. It is less important than the rest of Section 3–3, but this type of non-linear equation crops up occasionally. The method for solving it is the kind of thing you should know where to look up, but should not try to memorize.

READ Example 3 on pages K98–99.

Notes

(i) *line −6, page* K98. If $n > 0$, this step is only valid if $y(x) \neq 0$ for all x in the domain of y. This is the reason for the statement on line 2 of page K99, that the solution $y = 0$ has been "suppressed". The procedure may also "suppress" other solutions of the equation, as the example discussed on pages K93–94 shows (that example is a particular case of Bernoulli's equation).
(ii) *line 11, page* K99. The choice of c is restricted by the condition that $u = y^{-1}$ must not take the value 0 for any x in the domain of y.

Exercise

Find the general solution of

$$y' - Py = -y^2$$

where P is a positive number and $y: R^+ \longrightarrow R^+$.

(This solution is used in the next passage from K.)

Solution

$y' - Py = -y^2$ is a Bernoulli equation with $n = 2$. Following the prescription on page K98 we multiply by y^{-2} and then write the equation in terms of $u = y^{-1}$. Both steps are fully justified since the codomain of y does not include 0. The first of these steps gives

$$y^{-2}y' - Py^{-1} = -1$$

which is equivalent to $-u' - Pu = -1$, that is, to

$$u' + Pu = 1.$$

An integrating factor for this equation is e^{Px}; this gives

$$\frac{d}{dx}(u(x)e^{Px}) = (u'(x) + Pu(x))e^{Px} = e^{Px}.$$

Integration gives

$$u(x)e^{Px} = \frac{1}{P}e^{Px} + c,$$

so that

$$u(x) = \frac{1}{P} + ce^{-Px}$$

and (by the definition of u)

$$y(x) = \frac{1}{(1/P) + ce^{-Px}} = \frac{P}{1 + cPe^{-Px}} \qquad (x \in R^+).$$

To ensure that the codomain of y is R^+ as required, we must require $c \geqslant -1/P$.

4.3.4 Summary of Section 4.3

In this section we defined the terms

integrating factor	(page K97)	* * *
Bernoulli's equation	(page K98)	* *

Techniques

1. We have seen how to solve the following types of differential equation.

   (i) *Linear homogeneous and normal*

   $$(a_1(x)D + a_0(x))y = 0$$ * * *

   (ii) *Linear nonhomogeneous and normal*

   $$(a_1(x)D + a_0(x)y) = h$$ * * *

   (iii) *Bernoulli's equation*

   $$(a_1(x)D + a_0(x))y = hy^n$$ * *

2. We have also seen how to perform

   $$\int xe^{x^2}, \int e^{ax}, a \neq 0,$$

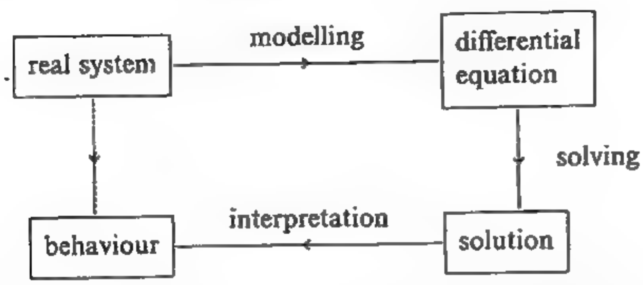
   $$\int xe^{ax}, \int \frac{y'}{y}.$$

   These results are recorded in **TI**.

4.4 SOME APPLICATIONS

4.4.0 Introduction

In this section we consider some examples illustrating how differential equations can be used to model real situations, and their solutions used to make quantitative statements or predictions about these situations. In these applications, solving the differential equation is only one part of the mathematician's work; he must also set up the differential equation to represent the real system he is studying, and interpret its solutions in terms of the behaviour of the real system.



We shall discuss other more complex modelling situations in *Unit 9, Differential Equations II*, *Unit 11, Differential Equations III*, and *Unit 13, Systems of Differential Equations.*

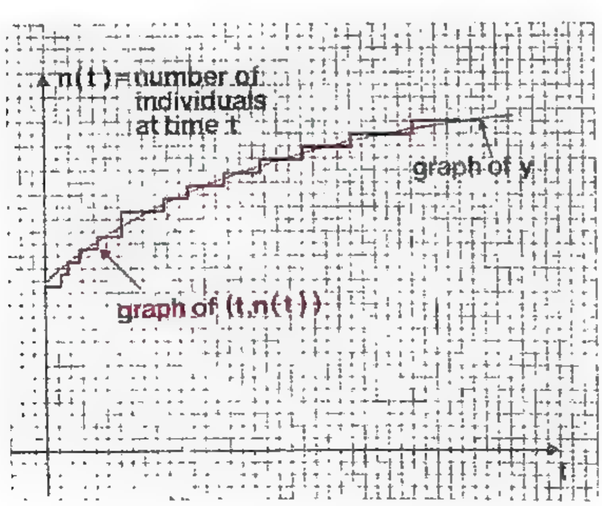
4.4.1 Growth and Decay

The application of first-order differential equations to population growth problems was discussed in the Foundation Course (*Unit M100 24, Differential Equations I*). The next reading passage covers roughly the same ground.

READ Section 4–9 *on page* K166, *to Figure 4-4 on page* K167.

Notes

(i) *line 15, page* K166. We are making the approximation that the number of individuals, a positive integer, can be adequately approximated by a continuously differentiable function y.



(ii) *line —1, page* K166. The notation here is not quite the same as in the solution in the previous section of this correspondence text: c here denotes the number denoted by cP there.

(iii) *line 4 and Figure 4-4, page* K167. From these mathematical results we can make predictions about the behaviour of any population that satisfies the dif-

ferential equation. The most obvious is that when t is very large, e^{-Pt} is very small and so y is very close to P; in other words

$$\lim_{t \text{ large}} y(t) = P.$$

Thus the population approaches the size that can just be supported by the available supply of the necessities of life.

Exercises

1. Exercise 9, page K169 (express your answer in terms of the relevant constant of proportionality).
2. Exercise 4, page K169.

Solutions

1. *Step 1: setting up the equation*

   Let the raindrop have radius $r(t)$ at time t. Then its volume is $\frac{4\pi}{3}[r(t)]^3$ and its surface area is $4\pi[r(t)]^2$.

   Evaporation causes a rate of *decrease of volume* proportional to its surface area, i.e.

   $$\frac{d}{dt}\left\{\frac{4\pi}{3}[r(t)]^3\right\} = -k \cdot 4\pi[r(t)]^2$$

   where $k > 0$ is a constant of proportionality. The equation can be written

   $$4\pi[r(t)]^2 r'(t) = -k4\pi[r(t)]^2$$

   or

   $$r'(t) = -k,$$

   provided $r(t) \neq 0$.

   Step 2: solving the equation

   It can be integrated at once (i.e. the integrating factor is 1) which gives

   $$r(t) = -kt + c.$$

   Step 3: interpreting the solution

   We are told that

   $$r(0) = r_0,$$

   i.e.

   $$0 + c = r_0$$

   and are asked to find t_1 such that

   $$r(t_1) = 0$$

   i.e.

   $$-kt_1 + c = 0.$$

   The first equation tells us that $c = r_0$ and the second then tells us that $t_1 = r_0/k$.

2. *Step 1: setting up the equations*

   The equation of growth is given:

   $$y' = (k_1 - k_2)y.$$

We must also consider the equation that would hold if there were no births, i.e. if $k_1 = 0$; this equation is

$$y' = -k_2 y.$$

Step 2: solving the equations

The general solution of the growth equation is

$$y(t) = ce^{(k_1 - k_2)t} \tag{1}$$

That of the no-births equation is

$$y(t) = de^{-k_2 t} \tag{2}$$

Step 3: interpreting the solution

We are told that the colony doubles in size every 24 hours; therefore, with t measured in hours, we must have (by Equation (1))

$$\frac{ce^{(k_1 - k_2)24}}{ce^{(k_1 - k_2)0}} = \frac{2 \text{ units of population}}{1 \text{ unit of population}}$$

therefore

$$e^{(k_1 - k_2)24} = 2$$

i.e.

$$k_1 - k_2 = \tfrac{1}{24} \ln 2$$

We are also told that if there were no births, the colony would halve in 8 hours, so that as above (by Equation (2))

$$e^{-k_2 8} = \tfrac{1}{2}$$

i.e.

$$k_2 = \tfrac{1}{8} \ln 2 \tag{3}$$

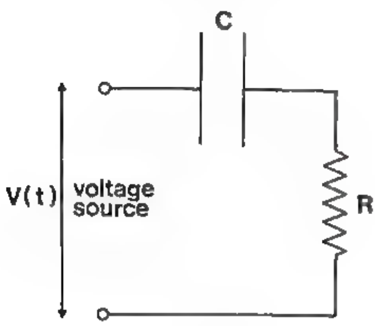
To find k_1 we substitute for k_2 in Equation (3), obtaining

$$k_1 = \tfrac{1}{6} \ln 2 = 0.12 \text{ (correct to two decimal places)}$$
$$k_2 = \tfrac{1}{8} \ln 2 = 0.09 \text{ (correct to two decimal places)}$$

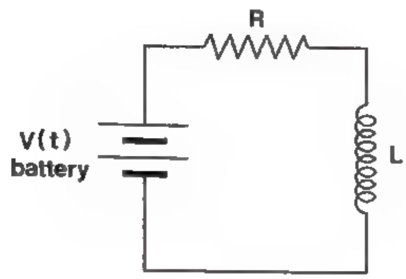
that is, the growth rate is about 12% per hour and the death rate is about 9% per hour.

4.4.2 Electric Circuits

The formulation of differential equations for electrical circuits containing resistances and condensers is treated in the television programmes of this and the previous unit. A summary of the rules is given at the top of page K170, which includes also the type of circuit element called an inductance. The next reading passage discusses a simple circuit containing a resistance and an inductance. The physics is different from the circuit considered in the television programme, but the mathematics is almost identical, as the following diagrams and equations show.



TV



K

$$R\frac{dq}{dt}+\frac{1}{C}q = V(t) \qquad L\frac{di}{dt}+Ri = V(t)$$

Equivalent roles are played by the following pairs of terms:

charge on condenser q : current i

resistance R: inductance L

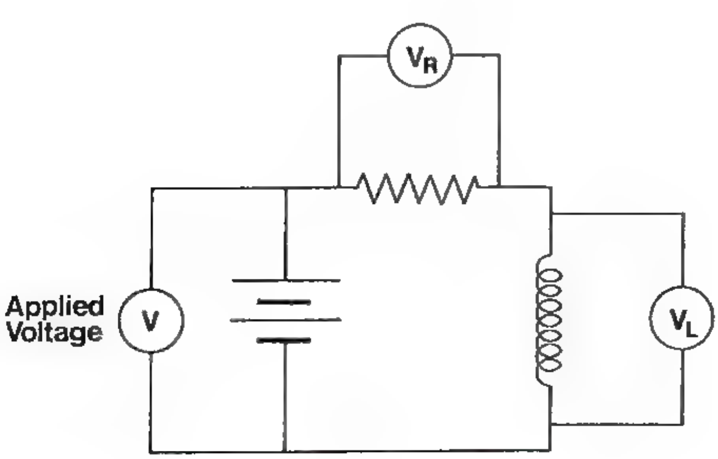
inverse capacitance $\frac{1}{C}$: resistance R

READ Section 4–10 on page K170 *to* "... term "impedance" here" *in the middle of page* K172.

Notes

(i) *law (b), page* K170. The "voltage drop" across an electrical component is the difference in electrical potential, as measured with a voltmeter. To law (b) we might add the rider that if a voltage source is applied to an *open* loop of the circuit then the sum of the voltages across the various components is equal to the applied voltage:

$$V = V_R + V_L$$



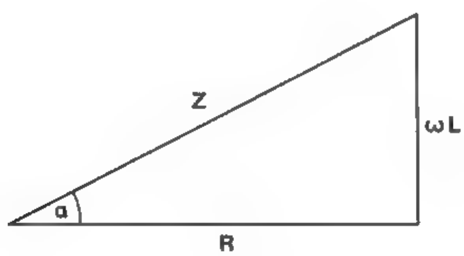
(ii) *line −3, page* K170. The "easy computation" is a particular case of Exercise 7 on page K99. A differential equation with a condition of the type

$$i(0) = \text{something}$$

is known as an *initial value problem.* These will be discussed in detail in *Unit 9, Differential Equations II.*
(iii) *line 8, page* K171. The "alternating current source" has a time-varying voltage of the form $E \sin \omega t$ (E constant). The differential equation is to be solved on the interval $I = [0, \infty)$.
(iv) *line 12, page* K171. The calculation leading to Equation (4-92) is given as an exercise below.
(v) *lines −9, −8, page* K171. The definitions of Z and α are shown in the diagram. Notice the use of the formula

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi,$$

with $\theta = \omega t$ and $\phi = -\alpha$.



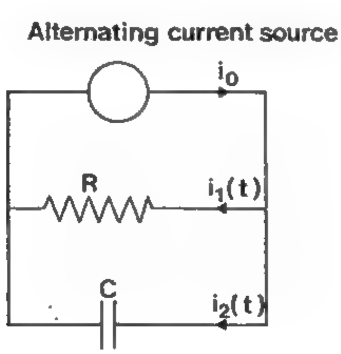
Exercises

1. Solve Equation (4-91), page K171 (take $t \in R_0^+$).
2. Exercise 3, page K175 (the voltage across a condenser is q/C where q is the charge, related to the current by $i = dq/dt$).

   Ignore the coulombs, microfarads, seconds, and ohms, but take

   $$C = 300 \times 10^{-6}, \text{ not } 300.$$

3. Set up the differential equation for the following circuit:



Solutions

1. Applying the method of Section 3–3 of **K** (pages K96–7) we rewrite Equation (4-91) as

   $$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \sin \omega t.$$

   An integrating factor is

   $$\exp \int (R/L)\, dt = \exp (R/L)t;$$

using it, we obtain

$$\left(\frac{di}{dt}+\frac{R}{L}i\right)\exp\frac{Rt}{L}=\frac{d}{dt}\left(i\exp\frac{Rt}{L}\right)$$

$$=\frac{E}{L}(\sin\omega t)\exp\frac{Rt}{L}.$$

Integrating both sides and using Section III.5.2 of TI we obtain

$$i\exp\frac{Rt}{L}=\frac{\frac{E}{L}\exp\frac{Rt}{L}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)}{\frac{R^2}{L^2}+\omega^2}+c$$

so that the general solution is

$$i=\frac{\frac{E}{L}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)}{\frac{R^2}{L^2}+\omega^2}+c\exp\left(-\frac{Rt}{L}\right).$$

To pick out the solution required, we determine c from the condition $i(0)=0$. The general solution gives

$$0=i(0)=\frac{-\frac{E}{L}\omega}{\left(\frac{R^2}{L^2}+\omega^2\right)}+c$$

so that we obtain

$$c=\frac{\frac{E}{L}\omega}{\left(\frac{R^2}{L^2}+\omega^2\right)};$$

substituting this into the general solution, we obtain

$$i=\frac{\frac{E}{L}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)}{\frac{R^2}{L^2}+\omega^2}+\frac{\frac{E}{L}\omega\exp\left(-\frac{Rt}{L}\right)}{\frac{R^2}{L^2}+\omega^2}$$

which is equivalent to the corrected version of Equation (4-92).

2. (a) *Step 1: setting up the equation*

Kirchoff's law (b) gives

$$iR+\frac{1}{C}q=0$$

and since $i=\frac{dq}{dt}$, the differential equation of the circuit (with the switch closed) is

$$R\frac{dq}{dt}+\frac{q}{C}=0.$$

Step 2: solving the equation

Separation of variables (see page K96) gives the general solution

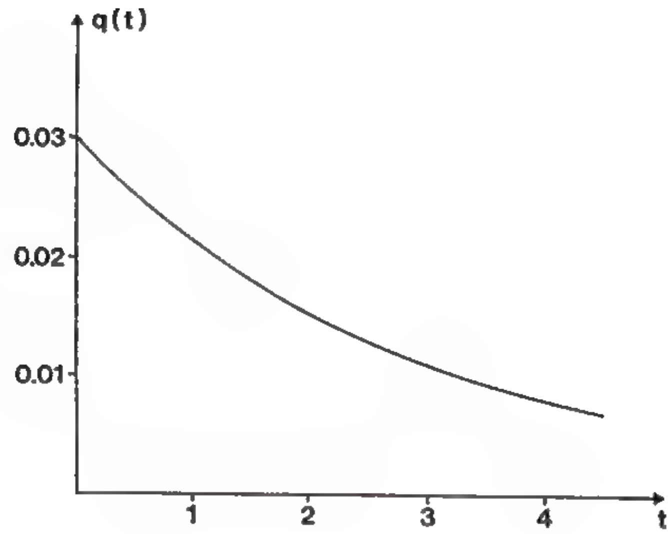
$$q = c \exp \frac{-t}{RC}.$$

Step 3: interpreting the solution

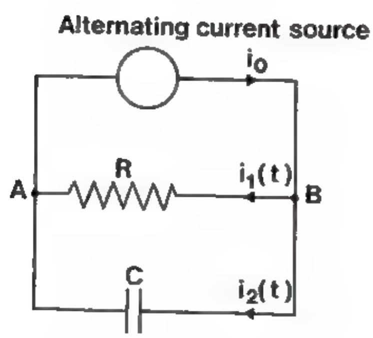
We are given $q = 0.03$ at $t = 0$ and so $c = 0.03$ and the required solution is

$$q = 0.03 \exp \frac{-t}{RC} = 0.03 \exp \frac{-t}{3}.$$

(b)



3.



Kirchoff's law applied to point A on the diagram gives

$$i_1(t) + i_2(t) = i_0 \qquad (1)$$

Kirchoff's second law applied to closed loop ABC gives

$$Ri_1(t) - Cq_2(t) = 0 \qquad (2)$$

where

$$i_2(t) = \frac{dq_2}{dt} \qquad (3)$$

We must reduce Equations (1) and (2) to a form which we can solve.

Differentiate Equation (2):

$$R\frac{di_1}{dt} - C\frac{dq_2}{dt} = 0$$

so, using Equation (3)

$$R\frac{di_1}{dt} - Ci_2(t) = 0.$$

Substitute into Equation (1) for $i_2(t)$:

$$R\frac{di_1}{dt} + Ci_1(t) = Ci_0.$$

Please note that for examination and assessment purposes you are *not* expected to be able to recall Kirchoff's laws and the form of voltage drops across network components. But, given the laws and the form of the voltage drops, you should be able to set up the appropriate differential equations.

4.4.3 Summary of Section 4.4

To analyse a physical situation in terms of a differential equation we follow the procedure: (page C35)

step 1: set up the equation
step 2: solve the equation
step 3: interpret the solution

Kirchoff's Laws enable us to set up the differential equations of an electrical network: (page K170)

Law (a): The algebraic sum of the currents flowing into any point in an electrical network is zero.
Law (b): The algebraic sum of the voltage drops across the various electrical components in an oriented closed loop of a network is zero.

The voltage drops across the electrical components considered in this section are given by (page K170)

q/C	for a condenser of capacitance C
iR	for a resistance R
$L\dfrac{di}{dt}$	for an inductance L

where q is the charge on the condenser and i is the current through the resistance or inductance.

The rate of accumulation of charge on a condenser is given by (page K172)

$$\frac{dq}{dt} = i$$

4.5 SUMMARY OF THE UNIT

We have looked at a particular infinite-dimensional vector space—the space of all real functions—and have chosen suitable subspaces of adequately differentiable functions. On these subspaces we have defined linear transformations of the form

$$a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_0(x).$$

Such linear transformations are called *linear differential operators*, and a linear problem involving such an operator is usually referred to as a *linear differential equation*. In the special case when the functions $a_n(x), \ldots, a_0(x)$ are constant functions, we have a *constant-coefficient* linear differential operator. (Their use will become apparent in *Unit 9, Differential Equations II.*)

The solution space of a linear homogeneous differential equation is finite-dimensional and its dimension equals the *order* of the equation. The terminology for linear differential equations is mostly coincident with that of the finite-dimensional linear problem which is given in *Unit 3, Hermite Normal Form.*

The normal first-order equation, which was solved in *Unit M100 24* is revised here. Applications which were considered include growth and decay of populations and carefully chosen electric circuits.

Definitions

The terms defined in this unit and page references to their definitions are given below.

continuously differentiable	(page K13)	* * *
interval	(page K3)	* * *
linear differential operator	(page K86)	* * *
order	(page K86)	* * *
linear differential equation	(page K91)	* * *
homogeneous	(page K91)	* * *
nonhomogeneous	(page K91)	* * *
normal	(page K91)	* * *
solution	(page K91)	* * *
null space (kernel)	(page K55)	* * *
general solution	(page K92)	* * *
particular solution	(page K92)	* * *
integrating factor	(page K97)	* * *
Bernoulli's equation	(page K98)	* *

Theorems

1. (Page K92)
   The dimension of the null space of L, a linear differential operator, equals the order of L. * * *

2. (Page K92)
   The solution set of a linear differential equation $Ly = h$ is * * *

   $$\{y: Ly = h\} = \{y_p\} + \{y: Ly = 0\}$$

   where $Ly_p = h$.

Techniques

1. Use $(aD + b)(cD + d) = acD^2 + (ad + bc)D + bd$ when a, b, c and d are constant functions. * * *

2. When a, b, c and d are not constant functions, apply the operator in 1 by applying each factor in turn. * * *

3. We have seen how to solve the following types of differential equation.

   (i) *Linear homogeneous and normal*

$$(a_1(x)D + a_0(x))y = 0$$ * * *

   (ii) *Linear nonhomogeneous and normal*

$$(a_1(x)D + a_0(x))y = h$$ * * *

   (iii) *Bernoulli's equation*

$$(a_1(x)D + a_0(x))y = hy^n$$ * *

4. We have also seen how to perform

$$\int xe^{x^2}, \int e^{ax}, a \neq 0,$$

$$\int xe^{ax}, \int \frac{y'}{y}.$$

   These results are recorded in **TI**.

5. To analyse a physical situation in terms of a differential equation we follow the procedure (page C35) * * *

   step 1: set up the equation
   step 2: solve the equation
   step 3: interpret the solution

Notation

$\mathcal{C}(I)$	(page K86)
$\mathcal{C}[a, b]$	(page K3)
D^2, D^n	(page K50)
$\mathcal{C}^n(I)$	(page K86)
(a, b)	(page C16)
(a, ∞)	(page C16)
$(-\infty, b)$	(page C16)
$(-\infty, \infty)$	(page C16)

4.6 SELF-ASSESSMENT

Self-assessment Test

This Self-assessment Test is designed to help you test quickly your understanding of the unit. It can also be used, together with the summary of the unit for revision. The answers to these questions will be found on the next non-facing page. We suggest you complete the whole test before looking at the answers.

1. Which of the following are linear differential operators? ($y \in C^\infty(I)$ in each case)

(a) $y \longmapsto y^4 + 3y^2 + xy$
(b) $y \longmapsto y^{(iv)} + 3y'' + xy$
(c) $y \longmapsto yy' + 2y$
(d) $y \longmapsto y' + 2y + 3$

2. Find $(xD)^2 f$, where $f(x) = x^3$ $(x \in R)$.

3. Which of (a), (b), (c), (d) and (e) correctly complete the following statement?

If F is the derived function of $G \in C^n(I)$, $(n \in Z^+)$, then F necessarily belongs to . . .

(a) $C^n(I)$
(b) $C^\infty(I)$
(c) $C^{n-1}(I)$
(d) $C(I)$
(e) $C^{n+1}(I)$

4. For each of the following equations, say whether it is

(a) homogeneous on I
(b) normal on I
(c) both
(d) neither

(i) $x^2y''(x) - xy'(x) + 3y(x) = 0$
with $I = (-3, 3)$

(ii) $x^2y''(x) - xy'(x) + 3y(x) = e^{-x}$
with $I = (3, \infty)$

(iii) $y''(x) + (\sin x)y(x) = x$
with $I = (-\pi, \pi)$

(iv) $y''(x) + (\sin x)y(x) = 0$
with $I = \left(0, \dfrac{\pi}{2}\right)$

(v) $|x|y''(x) = \sin x$
with $I = (-\pi, 0)$

5. For each of the following, say whether it is a solution of the equation:

$$(D^2 + D - 2)y(x) = 0 \qquad (x \in R)$$

(i) $y(x) = e^x$
(ii) $y(x) = e^{-x}$
(iii) $y(x) = e^{2x}$
(iv) $y(x) = e^{-2x}$

6. Write down the general solution of the equation in Question 5.

7. For each of the following, say whether it is a solution of the equation:

$$(D^2 + D - 2)y(x) = -2e^{-x} \qquad (x \in R)$$

(i) $y(x) = e^x$

(ii) $y(x) = e^{-x}$
(iii) $y(x) = e^{2x}$
(iv) $y(x) = e^{-2x}$

8. Write down the general solution of the equation in Question 7.

9. Find the integrating factor for each of the following equations:

   (i) $3y'(x) + 4y(x) = e^{-x} \qquad (x \in R)$
   (ii) $2xy'(x) - y(x) = x^3 \qquad (x \in R^+)$

10. Write down the general solution for each of the equations in Question 9.

11. What is the null space of each of the operators defined by

    (i) $L: y \longmapsto 3y' + 4y$
    and (ii) $L: y \longmapsto 2xy' - y,$

    the linear operators in Question 9?

12. By the substitution $u = y^2$, reduce

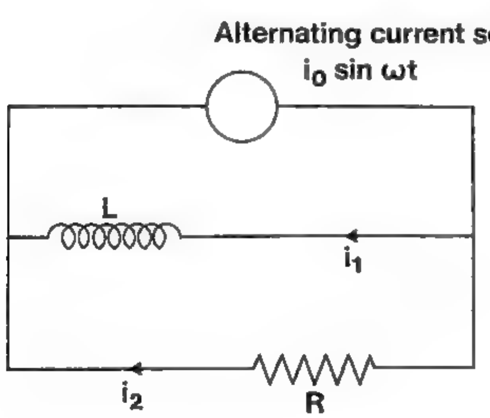
$$yy' + xy^2 - 2x = 0$$

    to a linear equation. State the linear equation.

13. For each of the following, state the order of the sum $L_1 + L_2$. All operators are defined on the interval $(-\pi, \pi)$.

    (i) $L_1 = 2xD + 3, L_2 = xD - 1$
    (ii) $L_1 = e^{-x}D^2 - D + 2$
    $L_2 = -e^{-x}D^2 + 2D + 2$
    (iii) $L_1 = D - 4, L_2 = \sin xD^2 - D + 1$

14. Write down a differential equation for the circuit below, introducing suitable notation.



Solutions to Self-assessment Test

1. (b) only.
   Compare (a), (b), (c) and (d) with Equation (3-1) on page K86.

2. $(xD)^2x^3 = (xD)(xDx^3) = (xD)(x3x^2)$
   $$= x \cdot 3 \cdot 3x^2$$
   $$= 9x^3$$

3. Only (c) and (d) correctly complete the statement. (See sub-section 1.2 of this correspondence text.)

4. (i) (a) (ii) (b) (iii) (b)
   (iv) (c) (v) (b).

5. (i) YES (ii) NO (iii) NO (iv) YES.
   (Check by substitution.)

6. $y = c_1e^x + c_2e^{-2x}$
   c_1, c_2 are arbitrary constants.

7. (i) NO (ii) YES (iii) NO (iv) NO

8. $y = e^{-x} + c_1e^x + c_2e^{-2x}$

9. (i) $e^{\int 4/3\,dx} = e^{4x/3}$ (ii) $e^{\int -1/2x\,dx} = x^{-1/2}$

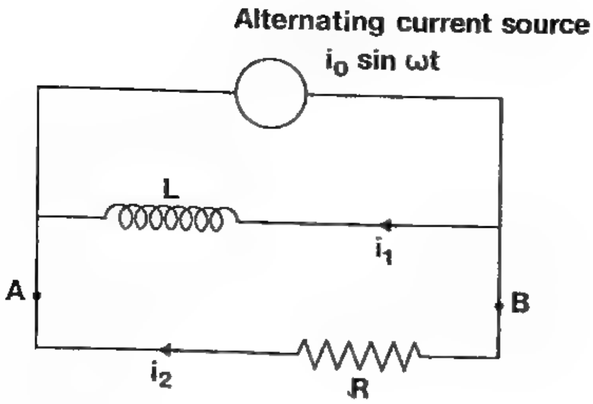
10. (i) Using the integrating factor of 9(i), the equation reduces to
   $$ye^{4x/3} = \tfrac{1}{3}\int e^{x/3}dx$$
   $$= e^{x/3} + c$$
   i.e.
   $$y = e^{-x} + ce^{-4x/3}.$$
   (ii) Using the integrating factor of 9 (ii), the equation reduces to
   $$yx^{-1/2} = \int \frac{x^{-1/2}x^2}{2}dx$$
   $$= \int \frac{x^{3/2}}{2}dx$$
   $$= \frac{x^{5/2}}{5} + c$$
   i.e.
   $$y = \frac{x^3}{5} + cx^{1/2}.$$

11. The null-space or kernel are those functions that map to the zero function under L.
   (i) From the solution to question 10(i), the null space is $\langle e^{-4x/3}\rangle$.
   (ii) From the solution to question 10(ii), the null space is $\langle x^{1/2}\rangle$.

12. If $u = y^2$, $u' = 2yy'$
   Hence
   $$\tfrac{1}{2}u' + xu - 2x = 0.$$

13. (i) First order
   (ii) First order ($L_1 + L_2 = D + 4$)
   (iii) Second order

14.



At a node

$$i_1 + i_2 = i_0 \sin \omega t$$

where i_0 and ω are constants.

Voltage between A and B is

$$i_2 R \qquad \text{(via } R\text{)}$$

$$L\frac{di_1}{dt} \qquad \text{(via } L\text{)}$$

Hence,

$$i_2 R = L\frac{di_1}{dt}.$$

So, eliminating i_2

$$L\frac{di_1}{dt} = R(i_0 \sin \omega t - i_1)$$

i.e.

$$L\frac{di_1}{dt} + Ri_1 = Ri_0 \sin \omega t.$$

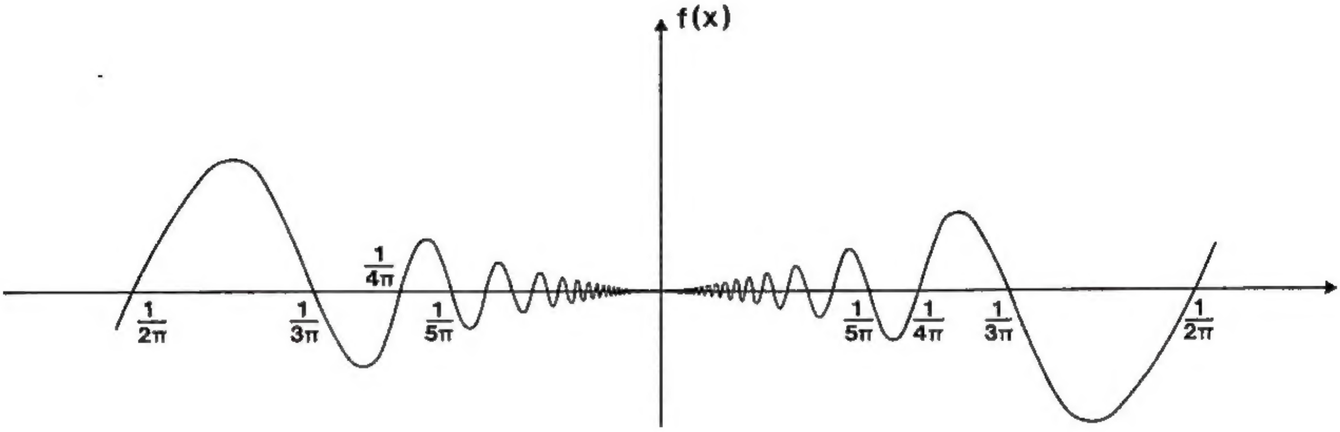
4.7 APPENDIX (OPTIONAL)

A differentiable function f, whose derived function is discontinuous yet has the same domain as f.

Let f be the function defined by

$$f: x \longmapsto \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad (x \in R)$$

Then f has the graph



This function is differentiable at every point in R and

$$Df: x \longmapsto \begin{cases} 2x \sin 1/x - \cos 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

But Df is *not* continuous at 0 : $Df(0) = 0$, but $\lim_{x \to 0} Df(x)$ does not exist.

First, we show that $Df(0) = 0$. We go back to first principles:

$$\begin{aligned} Df(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} \\ &= \lim_{h \to 0} \frac{h^2 \sin 1/h - 0}{h} \\ &= \lim_{h \to 0} h \sin 1/h \end{aligned}$$

Given $\varepsilon > 0$, consider $h \in [-\varepsilon, \varepsilon]$ $h \neq 0$.

Then,

$$\left| h \sin \frac{1}{h} \right| = |h| \times \left| \sin \frac{1}{h} \right| \leqslant |h|, \text{ since} \left| \sin \frac{1}{h} \right| \leqslant 1$$

i.e.

$$h \sin \frac{1}{h} \in [-\varepsilon, \varepsilon].$$

Therefore

$$\lim_{h \to 0} h \sin \frac{1}{h} = 0$$

so that

$$Df(0) = 0.$$

Next, we show that $Df = f'$ is not continuous at 0.

For f' to be *continuous* at $x = 0$, we would have to be able to choose, for any $\varepsilon > 0$, an interval $[-\delta, \delta]$ such that whenever x is in this interval, $|f'(x) - f'(0)| \leqslant \varepsilon$; that is

$$|f'(x)| \leqslant \varepsilon.$$

But

$$f'(x) = -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) \qquad (x \in R, x \neq 0)$$

and although the second term gets as near to zero as we please for, x close to zero, this is *not* true of the $-\cos\left(\frac{1}{x}\right)$ term, which oscillates between -1 and 1, as x gets closer to 0. So it is intuitively clear that this function is not continuous at 0. To be rigorous about it, we choose $\varepsilon = \frac{1}{2}$ (any value less than 1 would in fact do), and proceed to show that we then *cannot* choose an interval $[-\delta, \delta]$ with the required property. So let δ be *any* number greater than zero, and let us examine the interval $[-\delta, \delta]$. If we choose an integer N with $N\pi > 1/\delta$, then letting $x = \dfrac{1}{N\pi}$, we see that $x \in [-\delta, \delta]$, and

$$\begin{aligned} f'(x) &= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) \\ &= \pm 1 + 0. \end{aligned}$$

Therefore

$$|f'(x)| = 1 > \varepsilon = \tfrac{1}{2}$$

and f' is not continuous at 0.

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